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AN ELEMENTARY ALGEBRA

BY

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In this manual the author has endeavored to apply to an elementary course of Algebra the same general plan of treatment which was adopted in his Series of Arithmetics.

Every important subject is approached by means of simple suggestive questions, which lead the pupil directly to the appropriate definitions of terms, and to the principles involved in the subject. This is believed to be a great improvement upon the common plan of presenting definitions dogmatically, without any preliminary consideration of the things to be defined.

The plan of combining mental exercises with written work, which has proved valuable in the Arithmetics, is also an important feature of this book.

Among other characteristics of the work are the early introduction of equations, and the frequent use of practical problems which are designed to interest the learner at every stage of his progress.

Great care has been taken to combine clearness and conciseness of treatment with fullness of topics, so that, in a small compass, there is furnished a course of Algebra sufficiently comprehensive for Grammar-schools, High-schools and Academies.

Bearing in mind that the work is elementary in its scope, the author has left to the higher Algebra which is to form a part of this Series of Mathematical Text-books, the full discussion of many subjects that are too abstruse for the easy comprehension of beginners, and has purposely avoided the introduction of nice distinctions, which, though proper enough at a later stage in the course of study, would be likely to perplex the young student. He has preferred to use the mathematical terms *positive* and *negative* in the sense which general usage has established, rather than to attempt a limitation of their signification by applying them only to quantities to denote their essential relations, regardless of the algebraic signs which stand before them.

For the convenience of such learners as may desire to study only the most essential portions of Algebra, some topics have been excluded from the body of the work and placed in an Appendix.

The author has been led to the preparation of this Course of Mathematics by the popular and just demand for text-books which are brief and yet complete, and which are so graded and mutually consistent as to secure for the learner the greatest economy of time and labor.

He ventures to hope that the kind approval so widely bestowed upon his Series of Arithmetics may be extended to the work now presented to the public.

CONTENTS.

APPENDIX.

ELEMENTARY ALGEBRA.

SECTION I.

INTRODUCTION.

ARTICLE **1.**—1. Name some thing that can be measured. With what can it be measured?

2. What measures may be used in expressing the distance, or length, between two places? The area or surface of a farm? The capacity or volume of a room? The size of a block of stone? The time between two dates? The magnitude of a number?

3. What do you call every thing that can be measured or computed?

4. Is 6 a quantity? 5 miles? 4 bushels? Pride? Space? Redness? Length?

5. The quantity 6 apples contains how many single things? What is each single thing? The quantity 10 pounds contains how many single things? What is each single thing?

6. What is a single thing in 12 dollars? In 8 days? In 4 a's? In 3 x's? In 25?

7. What name is given to one of the things of which any quantity is composed?

8. What is the unit in 3 acres? In 5 dollars? In 11 z's? In 9 tens?

9. Express the number five by single marks; by a figure; by a Roman letter.

10. Express the number or quantity six by marks; by a figure; by Roman letters.

11. You have now expressed number or quantity in what ways?

12. In the Roman notation, what letter represents one? Five? Ten? Fifty? One hundred? One thousand?

13. Letters, then, may be used to express what? Five is frequently expressed by the letter V, but suppose we agree to express it by a, or b, or c, or any letter whatever; what then will be the value of the letter used?

14. If the value of a is five, what is the value of 2 a's? Of 3 a's?

15. If we agree to denote five apples by a and six apples by b, what will a plus b denote?

16. If we let a stand for seven, $2a$ for twice seven, $3a$ for three times seven, and so on, $10a$ will stand for what? $20a$? $4a$ plus $5a$? $8a$ plus $2a$? $12a$ less $7a$? $9a$ less $6a$?

17. John has 4 equal piles of apples, each pile containing 15 apples. How many times 15 apples has he?

18. If we denote 15 apples by b, how many times b will denote the entire quantity of John's apples? What, then, will denote the entire quantity?

19. What will denote the entire quantity if we denote the quantity in one pile by c? By d? By e?

20. From these examples what may you infer in regard to using the same letter to represent different quantities? In regard to using different letters to represent the same quantity?

21. George tells me that he has a certain number of cents, and that his sister has twice as many, but he does not tell me the number.

22. If I denote the unknown number of cents which George has by x, what will denote the number which his sister has? What, then, will denote how many they together have?

23. If $3x$ stand for the number they both have, which George at last tells me is 15, what does $1x$ stand for? $2x$? How many cents has George? How many has his sister?

24. I walked a certain distanee one day, twiee as far the second day, and three times as far the third day, and then found I had walked 60 miles. Tell me how far I walked the first day?

25. What quantity in this ease is uuknown to you? If you represent that unknown quantity by y, by what will you represent the distance I walked the second day? The third day? What will represent the entire distance?

26. If $6y$ stands for 60 miles, for what does y stand? $2y$? $3y$?

27. What letters are commonly used to express known quantities? To express unknown quantities?

Definitions.

2. Quantity is anything that admits of measurement or computation.

3. The **Unit** of a quantity is one of that quantity.

4. A **Symbol** of a quantity is a character used to express a quantity.

5. Known Quantities are expressed by figures or by the first letters of the alphabet.

Thus, quantities expressed by 1, 2, etc., or by a, b, etc., are understood to be those whose values are known or determined.

6. Unknown Quantities are expressed by the last letters of the alphabet.

Thus, quantities expressed by x, y, z, etc., are understood to be those whose values are unknown or undetermined.

7. Numerical Quantities are those expressed by figures.

8. Literal Quantities are those expressed by letters.

9. A **Sign** is a character used to express a process or to abbreviate an expression.

10. Algebra is the method of reasoning about quantities by means of symbols employed to express the quantities, and of signs employed to express their relations.

Signs.

11. The **Sign of Addition** is $+$, and is called *plus.* When placed between two quantities, it means that they are to be added.

Thus, $a+b$ is read "a plus b," or b added to a.

12. The **Sign of Subtraction** is —, and is called *minus.* When placed between two quantities, it means that the one on the right is to be taken from that on the left.

Thus, $a-b$ is read "a minus b," or b subtracted from a.

13. The **Sign of Multiplication** is $\times$, and is read *times*, or *multiplied by.* When placed between two quantities, it means that they are to be multiplied together.

Thus, $a\times b$ is read "b times a," or a multiplied by b.

The sign $\times$ is often omitted, and the multiplication is expressed by writing the letters together, the one after the other.

Thus, ab expresses the multiplication of a by b.

14. The **Sign of Division** is $\div$, and is read *divided by.* The dividend is placed at the left of the sign and the divisor at the right of it.

Thus, $a\div b$ is read "a divided by b."

Division may also be denoted by writing the dividend above and the divisor below a short horizontal line.

Thus, $\frac{a}{b}$ denotes the division of a by b.

15. The **Sign of Equality** is $=$, and is read *equals* or *equal.* When placed between expressions, it denotes that they are equal.

Thus, $a+b=x-y$, is read "a plus b equals x minus y."

16. The **Signs of Aggregation**, the *Parentheses,* ( ), the *Brackets,* [ ], and the *Vinculum,* ——, denote that the quantities included or connected by them are to be treated as a single quantity, subject to the operation indicated.

Thus, $(a+c)b$, $[a+c]b$, or $\overline{a+c}\times b$, each denotes that $a+c$ is to be multiplied by b.

Coefficients and Exponents.

17. A **Factor** of a quantity is any one of the multipliers used in producing that quantity.

Thus, a and b are factors of the quantity ab.

18. A **Coefficient** of a quantity is a number prefixed to the quantity to show how many times the quantity is taken.

Thus, in $5a$ the 5 is the coefficient of the quantity a, and shows that a is taken 5 times.

Numerical coefficients are expressed by figures, *literal coefficients* by letters, and *mixed coefficients* by both letters and figures.

Thus, in $5ab$ the 5 is the numerical coefficient of ab, and $5a$ is the mixed coefficient of b.

When no coefficient of a quantity is expressed, the coefficient 1 is understood.

Thus, a is the same as $1a$, and bx as $1bx$.

19. An **Exponent** of a quantity is a number written at the right and above the quantity, to denote how many times it is taken as a factor.

Thus, in a^3 the ^{3} is the exponent of the quantity a, and shows that a is taken 3 times as a factor; and in a^n, which is read a to the *nth*, or a, *nth*, the ^{n} shows that a is taken n times as a factor.

When no exponent of a quantity is expressed, the exponent ^{1} is understood.

Powers and Roots.

20. A **Power** of a quantity is the result obtained by taking that quantity one or more times as a factor.

Thus, a^2, which equals $a \times a$, is the square, or second power, of a; and a^3, which equals $a \times a \times a$, is the cube, or third power, of a.

When a quantity has no exponent expressed, the first power of the quantity is understood.

Thus, a is the first power of a.

21. A **Root** of a quantity is a quantity which, taken as a factor one or more times, will produce the quantity.

Thus, a is the second, or square, root of a^2, which equals $a \times a$, and the third, or cube, root of a^3, which equals $a \times a \times a$.

22. A **Root** of a quantity is denoted by the **Radical Sign,** $\sqrt{\ }$, or by a fractional exponent.

23. The number written in the opening of the radical sign is the **Index** of the root.

When the radical has no index expressed, the index 2 is understood.

Thus, $\sqrt{a}$, $\sqrt[2]{a}$, or $a^{\frac{1}{2}}$, denotes the second, or square, root of a, and $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, denotes the third, or cube, root of a.

Axioms.

24. Axioms are self-evident truths. Algebraic reasoning is based upon definitions and the following axioms:

1. *If equals be added to equals, the sums will be equal.*
2. *If equals be subtracted from equals, the remainders will be equal.*

3. *If equals be multiplied by equals, the products will be equal.*

4. *If equals be divided by equals, the quotients will be equal.*

5. *If a quantity be both increased and diminished by another quantity, the value of the former will not be changed.*

6. *If a quantity be both multiplied and divided by another quantity, the value of the former will not be changed.*

7. *Quantities which are equal to the same quantity are equal to each other.*

8. *The whole of a quantity is equal to the sum of all its parts.*

9. *Like powers and like roots of equal quantities are equal.*

EXERCISES.

25.—Ex. 1. $13+16=$ how many? *Ans.* 29.

2. $11+10+9=$ how many?

3. $17+13-8=$ how many? *Ans.* 22.

4. $19+6+11=$ how many?

5. $(18\times11)\div2=$ how many? *Ans.* 99.

6. What is the value of $38-(10+15)$?

7. What is the value of $\frac{(14+14)\times16}{4}$? *Ans.* 112.

8. What is the value of $\overline{97-31}\div22$?

9. What is the value of $\frac{13+9}{11}+\frac{24-10}{7}$? *Ans.* 4.

10. $5\times(8+3-6)=$ how many?

11. $(7-2)\times(10-3)=$ how many? *Ans.* 35.

12. What is the value of $8+21-2\times8-5$?

13. What is the value of $(14+6-8)\div(8-4)$? *Ans.* 3.

SECTION II.

ALGEBRAIC EXPRESSIONS.

26.—Ex. 1. If a express a quantity, what will express five times that quantity?

2. If a express one quantity and b another quantity, by what sign will you connect them to express their sum?

3. In the expression $a+b$, what are the parts connected? In $a-b$, what are the parts connected?

4. If $a=8$ and $b=3$, what is the value expressed by $a+b$? By $a-b$?

Definitions.

27. An **Algebraic Expression** is a quantity expressed in algebraic language.

Thus, $5a$ is the algebraic expression of 5 times the quantity denoted by a.

28. The **Terms** of an algebraic expression are the parts connected by the sign + or −.

Thus, ab and cb are the terms of $ab+cb$, and x and y the terms of $x-y$.

Terms are *positive* or *negative* according as they have the sign + or −.

Thus, in $ab-cb$, the term ab is positive, and the term $-cb$ is negative.

29. Similar Terms are those which contain the same letters having the same exponents.

Thus, $2xy^2$ and $-7xy^2$ are similar terms.

30. Dissimilar Terms are those which contain different letters or different exponents of the same letter.

Thus, $4a^2b$ and $3ac^2$ are dissimilar terms; also $4a^2b$ and $4ab^2$.

31. A **Monomial** is an algebraic expression consisting of one term.

Thus, ab^2, $-bx$, etc., are monomials.

32. A **Polynomial** is an algebraic expression consisting of more than one term.

Thus, $a-c$ and $a+ab^2-b$ are polynomials.

33. A **Binomial** is a polynomial consisting of two terms.

Thus, ab^2+cd and $xy-3x^2y$ are binomials.

34. A **Trinomial** is a polynomial consisting of three terms.

Thus, $ax+xy-ab$ and $cd+bx+b$ are trinomials.

35. The **Degree** of a term is the number of its literal factors.

Thus, $2x$, which contains only one literal factor, is of the first degree, and $6ab^2$, which contains three literal factors, is of the third degree.

36. The **Numerical Value** of an algebraic expression is the result obtained by substituting for its letters definite numerical values, and performing the processes denoted by the signs.

Thus, the numerical value of $6a+b^2-c$, when $a=2$, $b=5$ and $c=11$, is $6\times2+5\times5-11$, which equals 26.

EXERCISES.

37. Express algebraically—

1. The sum of x and y. *Ans.* $x+y$.

2. The value of 2 times a diminished by b.

3. Five times x, added to three times y, diminished by c times d. *Ans.* $5x+3y-cd$.

4. The difference of a and c multiplied by the sum of a and b. *Ans.* $(a-c)\ (a+b)$.

5. Two times a multiplied by b square, plus four times b multiplied by c cube. *Ans.* $2ab^2+4bc^3$.

6. a third power into b fourth power, plus two times a second power into c, minus three a second power into b third power.

7. Three times x plus y, divided by seven times the product of a multiplied by b. *Ans.* $\frac{3x+y}{7ad}$.

8. a minus b, divided by a plus b.

9. a minus b, multiplied by c into x. *Ans.* $(a-b)cx$.

10. The square root of $(a-c)$. *Ans.* $\sqrt{a-c}$.

11. The cube root of a, plus the fourth root of x plus y.
Ans. $\sqrt[3]{a}+\sqrt[4]{x+y}$.

12. Write a polynomial of three terms.

38. What are the numerical values of the following expressions, when $a=3$, $b=5$, $c=2$, $d=7$, $m=2$, and $n=3$?

1. $a+b-2c$. *Ans.* 4.
2. $2a+b^2+3b$.
3. $4a^2-b+5c$. *Ans.* 41.
4. $(a+b^3)c$. *Ans.* 256.
5. $(a+d)(b-c)$.
6. $cd(b+c)+ab$. *Ans.* 113.
7. $ad+\frac{a+b}{c^2}$.
8. $\frac{2a+d^2}{d+2c}+(m+n)b$. *Ans.* 30.
9. $c^2+4a^3-c^2a+b^2$.
10. If $x=6$ and $y=8$, what is the value of $2x+(x+y)x+15$? *Ans.* 111.
11. If $a=15$, $b=10$ and $c=25$, what is the value of $\sqrt{b^2}+\sqrt{4c}-2a$? *Ans.* -10.
12. If $x=11$, $y=9$ and $a=20$, what is the value of $(x+y)(x-y)+\sqrt{5a}$?
13. If $a=5$, $b=6$, $m=4$ and $n=10$, what is the value of $10a+b\sqrt[3]{1000n^3}-\sqrt{m^3}$? *Ans.* 642.
14. If $a=4$, $b=10$, $x=12$ and $y=16$, what is the value of $(a+b)(ab+8)+\sqrt[3]{x^3}-2y$?

SECTION III.

ALGEBRAIC PROCESSES.

39. A **Problem** is something to be done or a question to be solved.

40. **Algebraic Processes** are the means employed in the solution of problems whose conditions can be expressed in algebraic language.

PROBLEMS.

41.—Ex. 1. Henry has twice as many books as Arthur, and both together have 24. How many books has each?

SOLUTION. Let x equal Arthur's number, then 2 times x, or $2x$, will equal Henry's number, and $3x$ will equal the number both together have, which is 24.

If $3x$ equal 24, x, or the number Arthur has, must equal one-third of 24, or 8, and $2x$, the number Henry has, must equal 2 times 8, or 16.

$$x = \textit{Arthur's number};$$
$$2x = \textit{Henry's number}.$$
$$x + 2x = 24$$
$$3x = 24$$
$$x = 8, \textit{ Arthur's number.}$$
$$2x = 16, \textit{ Henry's number.}$$

2. A man is four times as old as his son, and the sum of their ages is 55 years. How old is each of them?

Ans. The man, 44 years; the son, 11 years.

3. A farmer gave for a farm and its stock \$3000, and the farm cost 5 times as much as its stock. What was the cost of each? *Ans. The farm, \$2500; its stock, \$500.*

4. The sum of two numbers is 200, and the larger is 3 times the smaller. What is the larger number?

5. My salary is \$3000 a year, and the portion of it I spend is 4 times the portion I save. What portion of it do I spend?

Ans. \$2400.

6. My horse and carriage are together worth \$750, and the carriage is worth twice as much as the horse. What is the value of each?

42.—Ex. 1. A plow, a harrow and a cart cost together \$96. The harrow cost twice as much as the plow, and the cart 5 times as much as the plow. What did each cost?

SOLUTION. Let x equal the cost of the plow; then $2x$ will equal the cost of the harrow, $5x$ the cost of the cart, and $8x$ the cost of all together, which is \$96.

If $8x$ equal \$96, x, or the cost of the plow, must equal one-eighth of \$96, or \$12; $2x$, or the cost of the harrow, must equal 2 times \$12, or \$24; and $5x$, or the cost of the cart, must equal 5 times \$12, or \$60.

$x =$ *the cost of the plow;*
$2x =$ *the cost of the harrow;*
$5x =$ *the cost of the cart.*

$$x + 2x + 5x = \$96$$
$$8x = \$96$$
$x = \$12$, *the cost of the plow.*
$2x = \$24$, *the cost of the harrow.*
$5x = \$60$, *the cost of the cart.*

2. The sum of three numbers is 180; the second is 5 times the first, and the third 6 times the first. What are the numbers? *Ans.* *15, 75, 90.*

3. Among three men, A, B and C, \$2000 were distributed, so that B and C each had twice as many dollars as A. How many dollars did each have?

4. The sum of the ages of three persons, A, B and C, is 96 years. B is twice as old as A, and C 3 times as old. How old are B and C each? *Ans.* *B, 32 years; C, 48 years.*

5. Divide \$600 among Smith, Downer and Herr, so that Downer shall have 3 times as much as Smith, and Herr twice as much as Downer. What will be the share of each?

6. A man has three farms, which together contain 320 acres, and the farms are in relative size as the numbers 1, 2 and 5. How many acres are there in each? *Ans.* *40, 80, 200.*

7. John has some marbles, George has four times as many as John, William has twice as many as John and George, and they all have 150 marbles. How many has each?

8. Andrew has 5 times as many cents as Joseph, and Edwin has one-half as many as Andrew and Joseph. If the three have in all 135 cents, how many has each?

43.—Ex. 1. John has 45 cents more than Willie, and 6 times Willie's number equals John's. How many has each?

SOLUTION. Let x equal Willie's number; then $6x$ will equal John's number, and $6x-x$ will be the number John has more than Willie, which is 45.

If $6x-x$, or $5x$, is equal to 45, x, or the number Willie has, must be one-fifth of 45, or 9, and $6x$, or the number John has, must be 6 times 9, or 54.

$$
\begin{aligned}
x &= \textit{Willie's number;}\\
6x &= \textit{John's number.}\\
\hline
6x-x &= 45\\
5x &= 45\\
x &= 9, \textit{ Willie's number.}\\
6x &= 54, \textit{ John's number.}
\end{aligned}
$$

2. The difference of two numbers is 40, and the larger is 5 times the smaller. What are the numbers? *Ans. 10 and 50.*

3. The difference between the ages of a mother and her daughter is 24 years, and the mother's age is 3 times that of the daughter. What is the age of each?

4. Five times the amount of my money diminished by three times that amount is equal to \$62. How much money have I? *Ans. \$31.*

5. Alice has 36 more books than Susan, and her number is 4 times Susan's number. How many books has each?

44.—Ex. 1. Alfred and Daniel have together a cents, and Daniel has 4 times as many as Alfred. How many cents has each?

SOLUTION. Let x equal Alfred's number; then $4x$ must equal Daniel's number, and $5x$ must equal the number both have together, which is a.

If $5x$ equals a, x, the number Alfred has, will equal one-fifth of a, or $\frac{a}{5}$; and $4x$, the number Daniel has, must equal 4 times $\frac{a}{5}$, or $\frac{4a}{5}$.

$$
\begin{aligned}
x &= \textit{Alfred's number;}\\
4x &= \textit{Daniel's number.}\\
\hline
x+4x &= a\\
5x &= a\\
x &= \frac{a}{5}, \textit{ Alfred's number.}\\
4x &= \frac{4a}{5}, \textit{ Daniel's number.}
\end{aligned}
$$

2. My horse and carriage cost me a, and the cost of the carriage was twice that of the horse. What was the cost of each?

3. The sum of three numbers is a, the second is 3 times the first, and the third is 4 times the first. What are the numbers?

4. The difference between two numbers is a, and the larger equals 7 times the smaller. What are the numbers?

Ans. $\frac{a}{6}, \frac{7a}{6}$.

5. Divide the number b into such parts that the larger part shall be 4 times the smaller.

6. Albert has a more cents than his brother, and his number is 3 times that of his brother. How many cents has each?

Ans. Albert, $\frac{3a}{2}$; *his brother,* $\frac{a}{2}$.

7. If a in the last problem equals 12, what is the value of the results?

8. Divide the number m into three parts that shall be to one another as 1, 2 and 3, and find the value of each, if m equals 60.

Ans. 1*st part,* $\frac{m}{6}=10$; 2*d part,* $\frac{2m}{6}=20$; 3*d part,* $\frac{3m}{6}=30$.

Test Questions.

45.—1. What is *Quantity?* The unit of a quantity? A symbol of a quantity? What are known quantities? Unknown quantities? Numerical quantities? Literal quantities? What is a sign? Algebra?

2. What is the *Sign of Addition?* Of subtraction? Of multiplication? Of division? Of equality? Of aggregation?

3. What is a *Factor* of a quantity? A coefficient? What coefficient is understood when no coefficient is expressed? What is an exponent? What exponent is understood when no exponent is expressed?

4. What is a *Power* of a quantity? A root of a quantity? How is a root of a quantity denoted?

5. What are *Axioms?* Upon what is Algebraic reasoning based?

6. What is an *Algebraic Expression?* What are the terms of an algebraie expression? What are similar terms? Dissimilar terms? What is a monomial? A polynomial? A binomial? A trinomial? The degree of a term? The numerieal value of an algebraie expression?

7. What are *Algebraic Processes?* What is a problem?

SECTION IV.

ADDITION.

46. Addition is the process of uniting two or more quantities to find their sum.

CASE I.

Terms Similar with like Signs.

47.—Ex. 1. A man earned once two dollars on one day, and twice two dollars on another day. How many times two dollars did he earn in the two days?

2. In one week John saves a certain sum of money, Harry saves twice that sum, and Robert saves three times that sum. How many times the sum do they together save? If x stands for the sum which John saves, what should stand for what they all save?

3. A merchant gained in January a certain amount, in February three times as much, and in March four times as much. How many times the gain in January was the entire gain? Let x represent the gain in January, what will represent the entire gain?

4. Once any quantity, plus three times that quantity, plus four times that quantity, is how many times that quantity?

5. Edward lost a number of cents, George lost four times as many, and Richard five times as many. How many times Edward's loss was the loss of the three boys? If $-x$ represent Edward's loss, what will represent George's loss? Richard's loss? The entire loss?

6. A grocer loses by one customer a sum represented by y, by another a sum five times as large, by another a sum seven times as large, and by another a sum twice as large. What will represent the whole loss? If y stands for ten dollars, what is the grocer's whole loss?

WRITTEN EXERCISES.

48.—Ex. 1. Henry has 2 apples, Alfred 3 apples and Arthur 9 apples. How many apples have they all?

SOLUTION. 2 apples, 3 apples and 9 apples are 14 apples. But let a denote 1 apple, then $2a$ will denote 2 apples, $3a$ three apples and $9a$ nine apples.
$2a$, $3a$ and $9a$ are $14a$, the apples they all have.

$$\begin{array}{r} 2a \\ 3a \\ \underline{9a} \\ 14a \end{array}$$

2. Henry has lost two apples, Afred 3 apples and Arthur 9 apples. How many apples have they all lost?

SOLUTION. Let $-a$ denote 1 apple lost; then, $-2a$ will denote 2 apples lost; $-3a$, 3 apples lost, and $-9a$, 9 apples lost.
$-2a$, $-3a$ and $-9a$ are $-14a$, the apples they all lost.

$$\begin{array}{r} -2a \\ -3a \\ \underline{-9a} \\ -14a \end{array}$$

3. What is the sum of $7a$, $6a$ and $11a$?

4. What is the sum of $-6bx$, $-10bx$ and $-13bx$?

49. Rule for Adding Similar Terms having like Signs.—*Add the coefficients, and to their sum prefix the common sign, and annex the common literal part.*

PROBLEMS.

(1.)	(2.)	(3.)	(4.)	(5.)
$3a$	$7bx$	$4y$	$-4x^2$	$-2bc^3$
$5a$	$5bx$	$5y$	$-x^2$	$-3bc^3$
$6a$	$4bx$	$3y$	$-3x^2$	$-2bc^3$
$7a$	bx	y	$-7x^2$	$-bc^3$
a	$2bx$	y	$-x^2$	$-bc^3$
$22a$			$-16x^2$	

(6.)	(7.)	(8.)	(9.)
$2ab^2x$	$-15ac$	$4x+3y$	$10ab-2cd$
$7ab^2x$	$-10ac$	$7x+4y$	$5ab-4cd$
$13ab^2x$	$-9ac$	$5x+2y$	$8ab-7cd$
ab^2x	$-ac$	$3x+7y$	$3ab-10cd$
$23ab^2x$		$19x+16y$	

10. What is the sum of bz, $5bz$ and $7bz$?

11. What is the sum of $5ac-d$, $3ac-7d$, $3ac-6d$ and $8ac-12d$?

12. Add $-3c(a-x)$, $-4c(a-x)$, $-7c(a-x)$, $-8c(a-x)$ and $-10c(a-x)$. *Ans.* $-32c(a-x)$.

CASE II.

Terms Similar with unlike Signs.

50.—Ex. 1. A man gained \$100 one day and lost \$50 the next day. What was the net result of the two days' transactions? If we give to gains the sign + and to losses the sign −, what will be the answer?

2. If a man gained \$50 one day and lost \$100 the next day, what was the result? If we give to gains the sign + and to losses the sign −, what will be the answer?

3. If John has 20 cents and owes 15 cents, what is his financial condition? What is it if he has 15 cents and owes 20 cents? If what one has in possession is regarded as positive and what one owes is made negative, what expression will show John's condition in the first case? In the second case?

4. In January a merchant gains a certain sum, in February he loses three times as much, and in March he gains six times as much. Let x stand for the gain in January; what will stand for the result of the three months' transactions?

5. Twice any quantity, plus four times the quantity, minus three times the quantity, minus five times the quantity, plus seven times the quantity, is how many times the quantity?

WRITTEN EXERCISES.

51.—Ex. 1. John earned in one week 8 dollars and spent 5 dollars, and the next week he earned 6 dollars and spent 7 dollars. How many dollars had he at the end of the second week?

SOLUTION. Let d denote a dollar earned and $-d$ a dollar spent; then, $8d$ and $6d$ denote the dollars earned, and $-5d$ and $-7d$ the dollars spent.
$8d$ and $6d$ are $14d$, $-7d$ and $-5d$ are $-12d$, and $-12d$ united with $14d$ cancels $12d$ of that quantity. Hence, their sum is $2d$, which shows that John had at the end of the week 2 dollars.

$$\begin{array}{r} 8d \\ -5d \\ 6d \\ -7d \\ \hline 2d \end{array}$$

2. A speculator gained at one time 200 dollars, at another lost 150 dollars, at another gained 300 dollars, and at another lost 400 dollars. What was the result of the transactions?

SOLUTION. Let d denote a dollar gained and $-d$ a dollar lost; then, $300d$ and $200d$ denote the dollars gained, and $-400d$ and $-150d$ the dollars lost.
$300d$ and $200d$ are $500d$, and $-400d$ and $-150d$ are $-550d$. $500d$ united with $-550d$ cancels $-500d$ of that quantity. Hence, their sum is $-50d$, which shows that the result of the transactions was a loss of 50 dollars.

$$\begin{array}{r} 200d \\ -150d \\ 300d \\ -400d \\ \hline -50d \end{array}$$

3. What is the sum of $3a$, $7a$, $-4a$, $-3a$ and $5a$?

4. What is the sum of $4ac$, $7ac$, $-6ac$ and $-2ac$?

52. Rule for Adding Similar Terms having unlike Signs.—*Add the coefficients of the positive and the negative terms separately. To the difference of these sums prefix the sign of the greater, and annex the common literal part.*

PROBLEMS.

(1.)	(2.)	(3.)	(4.)
$5a$	$7abx$	$-6ax^2$	$13xyz$
$7a$	$3abx$	$-5ax^2$	$-11xyz$
$-8a$	$-5abx$	$8ax^2$	$4xyz$
$-3a$	$-6abx$	ax^2	$3xyz$
a	$-2abx$	$-3ax^2$	$-12xyz$
$2a$		$-5ax^2$	

5. What is the sum of $3am^2$, $7am^2$, $-3am^2$, am^2 and $-9am^2$?

6. What is the sum of $8a^3b$, $-4a^3b$, a^3b, $9a^3b$ and $14a^3b$?

7. Add $6ac-d$, $4ac+3d$, $5ac-7d$, $4ac+2d$ and $-5ac+d$.
Ans. $14ac-2d$.

8 Add $10ab-4cd$, $3ab+2cd$, $-6ab+7cd$ and $12ab-9cd$.

9. Add $3x(a-c)$, $5x(a-c)$, $-3x(a-c)$, $-10x(a-c)$ and $4x(a-c)$.
Ans. $-x(a-c)$.

10. Add $4x^2-3y^2$, $2x^2-5y^2$, $-x^2+y^2$ and $2x^2+4y^2$.

11. Find the sum of $x^2+y^4+z^3$, $-4x^2-y^4+5z^3$, $8x^2-7y^4+10z^3$ and $x^2+6y^4-6z^3$.
Ans. $6x^2-y^4+10z^3$.

12. Find the sum of $x^3+(a-b)x^2+5x$, $14x^3-3(a-b)x^2-2x$ and $-5x^3+7(a-b)x^2-x$.
Ans. $10x^3+5(a-b)x^2+2x$.

CASE III.

Terms Similar or Dissimilar, with like or unlike Signs.

53.—Ex. 1. A earns one day a dollars, and another day b dollars. How many dollars does he earn in both days?

2. B gains a dollars on Monday and loses b dollars on Tuesday. What will represent the result of his two days' labor?

3. Peter has in the morning a cents. During the day he pays out and receives money as follows: Receives $2a$ cents, pays out b cents, receives a cents, pays out $3a$ cents, receives $4b$ cents, receives $5a$ cents, and pays out $7b$ cents. What will be his financial condition at night?

4. $3x+4y-2x-7y$, equals what?

WRITTEN EXERCISES.

54.—Ex. 1. What is the sum of $3a$, $7b$ and $-cd$?

SOLUTION. The sum $3a$ and $7b$ is $3a+7b$, and the sum of $3a+7b$ and $-cd$ is $3a+7b-cd$. Since the terms are dissimilar, the sum admits of no simpler form.

$$\begin{array}{r} 3a \\ 7b \\ -cd \\ \hline 3a+7b-cd \end{array}$$

2. What is the sum of $6a+8b$, $7a-4b+2ab$ and $2b-9ab$?

SOLUTION. For convenience in adding, similar terms are written in the same column.

Beginning with the column at the right to add, we find that $-9ab$ and $+2ab$ are $-7ab$; $2b$, $-4b$ and $8b$ are $6b$; and $7a$ and $6a$ are $13a$. Hence, the sum is $13a+6b-7ab$.

$$\begin{array}{r} 6a+8b \\ 7a-4b+2ab \\ 2b-9ab \\ \hline 13a+6b-7ab \end{array}$$

3. What is the sum of $4ab-x^2$, $3x^2-2ab$ and $2ax+2bx$?

4. What is the sum of $3a+x$, $4a+3b+2c$ and $2a-4b-c$?

This case evidently includes the two preceding cases, hence the following—

55. Rule for the Addition of Algebraic Quantities.—*Write similar terms in the same column, add each column, and connect the results with their proper signs.*

PROBLEMS.

(1.)

$$\begin{array}{l} 3ab+2bx^2 \\ -5bx^2-6ay^2 \\ -3bx^2 \\ \hline \end{array}$$

(2.)

$$\begin{array}{l} 2x^2+y^4+2z^3 \\ x^2-2y^4 \\ 8x^2+z^3 \\ \hline \end{array}$$

3. Add $x^3-2ax^2+a^2x+a^3$, x^3+3ax^2 and $2a^3-ax^2-2x^3$.

4. Add $3a^2+5ac-2b^3$, $-2a^3-4ab-b^3$ and $5a^3-4ac+7ab$.

5. What is the sum of $3a^2+4\sqrt{x}$, $5a-2\sqrt{x}$, $5a^2+12$ and $b+3\sqrt{x}+7$? *Ans.* $8a^2+5a+5\sqrt{x}+19+b$.

6. What is the sum of $3a+2b-5$, $a+5b-c$ and $3a-2c+3$?

7. Add a^2+ab+b^2-c, $3a^2-7b^2$, $4a^2+5ab$ and $-3ab-3b^2+c$.

8. Add $7x+3y+8z-4$, $5z-7+3x-8y$, $3y-5x+6-2x$ and $2-4x+3y-2z$.

9. Add a^2+ax+x^2, $3a^2-4ax+2x^2$ and a^2+x^2+a+x.

Ans. $5a^2-3ax+4x^2+a+x$.

10. Add $4a+b-3(a-x)$, $3a+7b-3c$, $3b+7(a-x)+9c-4$, $4(a-x)-2a+7$ and $a+2(a-x)+b+c-3$.

11. Add $13a^3(x+5y)+21$, $-10a^3(x+5y)+16a$, $-3a^3(x+5y)-8$ and $-13-8a$.

12. What is the sum of $7x^2-16+5y$, $24-\sqrt{xy}-15x^2+3xy$ and $3x^2-14+28y$? *Ans.* $-5x^2+3xy-\sqrt{xy}+33y-6$.

13. Add $6a(4x+7y)+14ab^2$, $9ab^2-cd+by$, $17a(4x+7y)-5by$, $10by-23ab^2+2cd$ and $-a(4x+7y)-cd$.

Ans. $22a(4x+7y)+6by$.

56. Dissimilar Terms having a common factor may be added by enclosing in a parenthesis the sum of the other factors, and annexing the common factor.

1. What is the sum of $2ax-bx+5cx$?

SOLUTION. $2ax$ is equal to $(2a)x$, $-bx$ is equal to $(-b)x$, and $5cx$ is equal to $(5c)x$. Hence, their sum is $(2a-b+5c)x$.

$$\begin{array}{r} 2ax \\ -\ bx \\ 5cx \\ \hline (2a-b+5c)x \end{array}$$

2. What is the sum of ay^2, by^2, $-acy^2$ and $2xy^2$?

3. What is the sum of $(a+b)x$ and cx?

4. What is the sum of xy^2+ay^2, $-3y^2$ and $-5ay^2$?

Ans. $(x-4a-3)y^2$.

5. What is the sum of $4ax-c$ and $b+dx$?

6. Add $(a-b)+x$, $c(a-b)-2y$ and $2x+y$.

Ans. $(1+c)(a-b)+3x-y$.

7 Find the sum of $(a-3b+c)x$ and $4ax+3bx$.

8. Find the sum of $3(a-b)$, $ax-bx$ and $-xy$.

9. Find the sum of $abx+nx$, $-dx^2$ and cx^2+dm.

Ans. $(ab+n)x+(c-d)x^2+dm$.

SECTION V.

SUBTRACTION.

57. Subtraction is the process of taking one quantity from another, or of finding the difference between two quantities.

The quantity from which the subtraction is made is called the **Minuend,** and the quantity which is subtracted is called the **Subtrahend.**

CASE I.

All the Terms Positive.

58.—Ex. 1. A gains 6 dollars and B gains 4 dollars; how much better off is A than B?

2. If John pays 3 cents for an orange and sells it for 5 cents, and we give gains the sign +, what will represent his gain?

3. A gains 6 dollars and B gains 4 dollars; how much worse off is B than A?

4. If a boy earns $3a$ dollars and spends $5a$ dollars, and we give losses the sign −, what will represent his financial condition?

WRITTEN EXERCISES.

59.—Ex. 1. From 11 apples take 7 apples.

SOLUTION.—Let a denote 1 apple; then $11a$ will denote 11 apples, and $7a$ will denote 7 apples. The difference between $11a$ and $7a$ is $4a$, or $11a-7a=4a$. This is the same as changing the sign of the subtrahend $7a$, making it $-7a$, and then adding it to the $11a$.

$$\begin{array}{r} 11a \\ 7a \\ \hline 4a \end{array}$$

or, $11a-7a=4a$

2. From $7a$ take $11a$.

SOLUTION. Taking from $7a$ all we can of $11a$, there remains $4a$ to be subtracted, or $-4a$. This is the same as changing the sign of the subtrahend $11a$, making it $-11a$, and then adding it to the $7a$.

$$\begin{array}{r} 7a \\ 11a \\ \hline -4a \end{array}$$

or, $7a-11a=-4a$

3. From a take $b+c$.

SOLUTION. Taking b from a, we have $a-b$, and taking also c, we have $a-b-c$; or $a-(b+c)=a-b-c$, in which, on removing the parenthesis, the signs of the subtrahend are changed from + to −

$$\begin{array}{l} a \\ \underline{b+c} \\ a-b-c \end{array}$$

or,

$$a-(b+c)=a-b-c$$

4. From $9x$ take $10x$. *Ans.* $-x$.

5. From $13bc$ take $11bc$.

6. From $14m^2n^2$ take $16m^2n^2$. *Ans.* $-2mn^2$.

7. From $4a$ take $3a+2b$. *Ans.* $a-2b$.

8. From a^2c+b^2c take $5b^2c$.

9. From $7y^2+3z$ take $5y^2+5z$. *Ans.* $2y^2-2z$.

10. From ax^2+bx^2 take $ax^2+3bc+bx^2$.

11. From $7a+2ac$ take $6a+9ac+b$. *Ans.* $a-7ac-b$.

12. From $3ab+2ab^2+ac$ take $2ab+3ab^2+9$.

Ans. $ab-ab^2+ac-9$.

CASE II.

One or more of the Terms Negative.

60.—Ex. 1. A loses 6 dollars and B loses 4 dollars; how much worse off is B than A?

2. John loses 6 dollars and Ray loses 4 dollars; if we give losses the sign −, how can we represent how much John is worse off than Ray?

3. A loses 6 dollars and B loses 4 dollars; how much better off is B than A?

4. A gains 6 dollars and B loses 4 dollars; how much better off is A than B?

5. What is the difference between 6 dollars loss and 4 dollars loss? Between 6 dollars gain and 4 dollars loss?

6. A gains 6 dollars and B loses 4 dollars; how much worse off is B than A? What is the difference between 6 dollars gain and 4 dollars loss?

7. A loses $6a$ dollars and B gains $4a$ dollars; how much worse off is A than B? If we give losses the sign $-$, how can we represent how much A is worse off than B?

8. A loses $6a$ dollars and B gains $4a$ dollars; how much better off is B than A? If we give gains the sign $+$, how can we express how much B is better off than A?

WRITTEN EXERCISES.

61.—Ex. 1. A ship at one time was found to be in 9 degrees north latitude, and at another time in 3 degrees south latitude. What is the difference of latitude between the two positions?

SOLUTION. Let d denote one degree of north latitude, and $-d$ a degree of south latitude, and we have the difference between the two positions evidently $9d+3d=12d$. This is the same as changing the sign of the subtrahend $-3d$, making it $3d$, and then adding it to the $9d$.

$$\begin{array}{r} 9d \\ -3d \\ \hline 12d \end{array}$$

or,

$$9d+3d=12d$$

2. From $-3a$ take $-5a$.

SOLUTION. $-3a$ is evidently equal to $2a-5a$, and if we take away the $5a$ there remains $2a$. This is the same as changing the sign of the subtrahend $-5a$, and adding it to the $-3a$.

$$\begin{array}{r} -3a \\ -5a \\ \hline 2a \end{array}$$

or,

$$-3a+5a=2a$$

3. From a take $b-c$.

SOLUTION. Taking b from a, we have $a-b$, a difference too small by c, since $b-c$ is to be taken from a; hence, the true difference must be $a-b+c$. Or, $a-(b-c)=a-b+c$, in which, on removing the parenthesis, the signs of the subtrahend are changed, $+$ to $-$ and $-$ to $+$.

$$\begin{array}{r} a \\ b-c \\ \hline a-b+c \end{array}$$

or,

$$a-(b-c)=a-b+c$$

4. From $a+c$ take $a-c$. *Ans.* $2c$.

5. From $-8a$ take $-3a$. *Ans.* $-5a$.

6. From $5x$ take $-2x+y$.

7. From $-7y$ take $9y$. *Ans.* $-16y$.

8. From $9xy$ take $-5xy$.

9. From $2by-x$ take $x-y$.

62. Rule for Subtraction of Algebraic Quantities.—*Conceive the signs of the subtrahend to be changed from + to −, or from − to +, and then proceed as in addition.*

PROBLEMS.

1. From $7a+3$ take $-11a$. *Ans.* $18a+3$.

2. From $a-b$ take $a+b$.

3. From x take $x-y$. *Ans.* y.

4. From $-48xy$ take $16xy$.

5. From $a+b-c$ take $a-b+c$.

6. From $a+8$ take $b-5$. *Ans.* $a-b+13$.

7. From $9(x+y)$ take $7(x+y)$. *Ans.* $2(x+y)$.

8. From $13ab^3$ take $-17ab^3$.

9. From $2(a-b)$ take $5(a-b)$. *Ans.* $-3(a-b)$.

10. From $5(a+b)$ take $-7(a+b)$.

11. From $7a+14b$ take $4a-10b$.

12. From $6a-2b-c$ take $2a-2b-3c$. *Ans.* $4a+2c$.

13. From $2a+3b-c$ take $a-2b-3c$.

14. From $3a-2b+3c$ take $2a-7b-c-d$.

15. From $7x^2-8x-1$ take $5x^2-6x+3$. *Ans.* $2x^2-2x-4$.

16. Subtract $-14(x+y-3)$ from $-7(x+y-3)$.

17. Find the value of $6x^2(a-d)-9x^2(a-d)+x^2(a-d)$.
Ans. $-2x^2(a-d)$.

18. Subtract $-a^3+3a^2b-3ab^2+b^3$ from $a^3-3a^2b+3ab^2-b^3$.

19. Subtract $2x-3a+8b+7c$ from $12x-5d+4b-3a$.
Ans. $10x-4b-7c-5d$.

20. From $5a(x^2+y^2)-b(x^2-y^2)+4c^2$ take $3a(x^2+y^2)-3b(x^2-y^2)$.

21. Subtract $4x^2y-7xy^2+5xz^2-24yz^2-xyz$ from $4x^2y-7x^2z+12y^2z-24yz^2+xyz$. *Ans.* $7xy^2-5xz^2-7x^2z+2xyz+12y^2z$.

63. Dissimilar Terms having a common factor may have their difference expressed by enclosing in a parenthesis the difference of their other factors, and annexing the common factor.

1. What is the difference between ax and bx?

SOLUTION. ax is equal to a times x, and bx is equal to b times x. Hence, $ax-bx$ is equal to $(a-b)$ times x, or to $(a-b)x$

$$\begin{array}{r} ax \\ bx \\ \hline (a-b)x \end{array}$$

2. What is the difference between ax^3 and cx^3?

3. What is the difference between $5x^2y^2z^2$ and $3xyz^2$?
Ans. $(5x^2y^2-3xy)z^2$.

4. What is the difference between ay^2 and cy^2+dy^2?
Ans. $(a-c-d)y^2$.

5. Subtract $xy+yz$ from $a-x^2+bx^2$.

6. Find the difference between $bx+c$ and $ax+d$.
Ans. $(b-a)x+c-d$.

7. Find the difference between $(3a-3x)cd$ and $5acd-3cdx$.
Ans. $-2acd$.

8. From $ax^2+2bx+ac^2$ take $bx^2-cx+2bc^2$.
Ans. $(a-b)x^2+(2b+c)x+(a-2b)c^2$.

Indicated Subtraction.

64. **A Parenthesis** indicates that all quantities enclosed by it are equally affected by some other quantity.

Hence, the subtraction of any quantity may be indicated by enclosing the quantity in a parenthesis, with $-$ prefixed, and writing the expression after the minuend.

The subtraction of $a-b+c$ from $c+d$ is indicated by enclosing the first quantity in a parenthesis, thus $c+d-(a-b+c)$.

Performing the indicated operation, we have $c+d-a+b-c$.

This expression may be transformed into the former expression, of equivalent value, by changing the signs of the last three terms, enclosing the terms in a parenthesis and prefixing the sign $-$.

Thus, $c+d-a+b-c=c+d-(a-b+c)$.

65. Principles.—1. *When an expression within a parenthesis is preceded by* $-$, *the parenthesis may be removed if the sign of every term within the parenthesis be changed.*

2. *Any number of terms in an expression may be enclosed in a parenthesis preceded by* $-$, *provided the signs of the terms are changed.*

PROBLEMS.

1. Indicate the subtraction of $5a-4x$ from $7bc$.
Ans. $7bc-(5a-4x)$.

2. Indicate the subtraction of $3+2a^2b$ from $3d-a^2b$.

3. What is the value of $5x+y-(x-2y)$? *Ans.* $4x+3y$.

4. What is the value of $4a-2b+6-(3a-7b-3)$?

5. Place the last two terms of $a-c-d$ in a parenthesis.
Ans. $a-(c+d)$, or $a+(-c-d)$.

6. Indicate the subtraction of $3y+4b^2-5c-7$ from ax.

7. Express in the simplest form the value of $7y+z-(x+y+2z)$.

8. What is the value of $3abc^3-3b^2+(b^2-2abc^3-cd^2)$?

9. Place in a parenthesis preceded by $-$ the last three terms of $2x^2+2y^3+3ax-13$. *Ans.* $2x^2-(-2y^3-3ax+13)$.

Test Questions.

66.—1. What is *Addition?* What is the rule for adding similar terms with like signs? For adding similar terms with unlike signs? For addition of algebraic quantities? How may dissimilar terms having a common factor be added?

2. What is *Subtraction?* What is the subtrahend? The minuend? The difference? What is the rule for subtraction of algebraic quantities? How may dissimilar terms having a common factor have their difference expressed?

3. What does a *Parenthesis* indicate? How may the subtraction of any quantity be expressed? When a parenthesis is preceded by minus, how may it be removed from the expression? What change of signs must be made when any number of terms of an expression are inclosed in a parenthesis preceded by the minus sign?

SECTION VI.

MULTIPLICATION.

67. Multiplication is the process of taking one of two quantities as many times as there are units in the other.

The quantity taken is called the **Multiplicand,** the quantity which shows how many times the multiplicand is taken is called the **Multiplier,** and the result is called the **Product.**

The multiplier and multiplicand are called **Factors** of the product.

EXERCISES.

68.—Ex. 1. If a man earns 5 dollars a day, what will he earn in 3 days?

2. If a man spends 4 dollars a day, what will he spend in 3 days?

3. If a man earns a dollars a day, and the sign + is given to that which he earns, what will express his earnings for 3 days?

4. If a man spends a dollars a day, and the sign $-$ is given to that which he spends, what will express his spendings for 4 days?

5. Let x represent a man's gains; what will represent six times his gains?

6. Six times the quantity $+x$ is how much?

7. Let $-x$ represent a man's losses; what will represent six times his losses?

8. Six times the quantity $-x$ is how much?

9. If y stand for what a man earns in one day, what will stand for his earnings in a days? In bc days?

10. If $-y$ represent a person's daily expenses, what will represent his expenses for 4 days? For a days?

11. The result of taking a plus quantity $+a$ times has what sign? The result of taking a minus quantity $+a$ times has what sign?

12. How much is 3 times $+5$? 4 times -7? a times plus x? b times minus y?

13. A plus quantity multiplied by a plus quantity gives what kind of a quantity? A minus quantity multiplied by a plus quantity gives what kind of a quantity?

14. If you take 3 times 4 dollars and subtract the product, how will you express the result?

15. What result is obtained by subtracting the product of 5 times 7 apples? 6 times a dollars? a times z units?

16. What is the result of $+6\times(-4)$, if it means that plus 6 is to be taken 4 times subtractively?

17. A plus quantity multiplied by a minus quantity gives what kind of a quantity?

18. If a person's income is expressed with the sign $+$, what sign shall be given to his outgoes? If gains are plus, what are losses? If earnings are plus, what are expenses?

19. A's income is \$1000; his outgoes, \$400. How will you express his net income? If from this expression you take away $-\$400$, the outgoes, what will be the net income?

20. A boy earns 25 cents and spends 15 cents. Express the amount saved. If he does away with his spending, the expression 25 cents $-$ 15 cents will then become what? Taking away -15 cents from the expression 25 cents -15 cents, is equivalent to adding how much to that expression?

21. I agreed to pay a boy 12 cents an hour for 7 hours' work, but I find that he played 3 hours out of the 7. How much ought I to pay him?

22. If -12 stands for the sum paid for 1 hour's work, what will stand for the sum payable for 7 hours' work? If the boy worked not 7 hours, but $7-3$ hours, -12 has been taken how many times too many? Then from -84 we must take what? Taking -36 from -84 is equivalent to adding what to -84?

23. What is the result of $-6\times(-4)$, if it means that -6 is to be taken 4 times subtractively?

24. What is the result of $-7\times(-5)$, if it means that -7 is to be taken 5 times subtractively?

25. If the $-$ sign of a quantity becomes $+$ when the quantity is subtracted, a minus quantity multiplied by a minus quantity gives what?

69. Principles.—1. *The coefficient of the product is equal to the product of the coefficients of the factors.*

Thus, since the factors of a product may be taken in any order, the product of $3a$ by $2b$ is the same as $3\times2\times a\times b$, or $6\times ab$, which is $6ab$.

2. *The exponent of a letter in the product is equal to the sum of its exponents in the factors.*

Thus, since the exponent of a letter denotes the number of times the letter is taken as a factor, the product of a^3 by a^2 is the same as $aaa\times aa$, or $aaaaa$, which is a^5.

3. *Two factors having like signs give a plus product, and two factors having unlike signs give a minus product.*

Thus, since a positive multiplier denotes that the multiplicand is to be taken additively, the product of $+a$ by $+b$ is found by taking $+a$ as many times as there are units in b, and adding the result, which gives $+ab$; and since a negative multiplier denotes that the multiplicand is to be taken subtractively, the product of $-a$ by $-b$ is found by taking $-a$ as many times as there are units in b, and subtracting the result, which gives $+ab$.

A positive multiplicand taken subtractively gives a negative product, as $+a\times(-b)$ gives $-ab$; and a negative multiplicand taken additively gives a negative product, as $-a\times(+b)=-ab$.

CASE 1.

A Monomial by a Monomial.

70.—Ex. 1. What is the cost of $2b$ tons of coal at $3a$ dollars per ton?

SOLUTION. The cost must be $2b$ times $3a$ dollars. $2b$ times $3a$ is the same as $3\times2\times a\times b$, or $6\times ab$, which is $6ab$.

$$\begin{array}{r} 3a \\ 2b \\ \hline 6ab \end{array}$$

2. Multiply $3a^3$ by $2a^2$.

SOLUTION. The product of $3a^3$ by $2a^2$ is the same as $3\times2\times a^3\times a^2$, which, by principles (Art. 69), is $6\times a^5$, or $6a^5$.

$$\begin{array}{r} 3a^3 \\ 2a^2 \\ \hline 6a^5 \end{array}$$

3. What is the product of $-5abc$ by $3bc$?

4. What is the product of $-7ax^2$ by $-3bx$?

71. Rule for Multiplying a Monomial by a Monomial.—*Find the product of the numerical coefficients of the two factors; annex to this product the different letters of both factors, giving to each letter an exponent equal to the sum of its exponents in the two factors, and make the result positive when the factors have like signs, and negative when they have unlike signs.*

PROBLEMS.

(1.)	(2.)	(3.)	(4.)
$3ab$	$-6x^3$	$9a^2b^m$	$-3xy^2$
ab^2	$3x^2$	$-a\,b$	$-2x^3y$
$3a^2b^3$	$-18x^5$	$-9a^3b^{m+1}$	$6x^4y^3$

(5.)	(6.)	(7.)	(8.)
$-2mn$	$8bdc^2$	$-6a^2x^m$	$11bdc^3$
mn^2	$2ab^2c^2$	$-2c^3x^n$	$-3bdc^2$

9. Multiply $-8bd^2$ by $-5c^2d$. *Ans.* $40bc^2d^3$.

10. Multiply $6b^nc^m$ by $3b^nc$. *Ans.* $18b^{2n}c^{m+1}$.

11. Multiply $2ac^3x^2$ by $-5acx^2$.

12. Multiply $-xy^m$ by $-xy^n$.

13. Multiply $-7a^2b$ by $3ab^2$. *Ans.* $-21a^3b^3$.

14. Multiply $-2pn$ by $-3cn$.

15. Multiply $-24ab^nd$ by $9abx$. *Ans.* $-216a^2b^{n+1}dx$.

16. Multiply $15a^2b^3y$ by $-6ax^3y$. *Ans.* $-90a^3b^3x^3y^2$.

17. Multiply x^n by x^{2n}.

18. Multiply $a(c+d)^2$ by $b(c+d)^3$. *Ans.* $ab(c+d)^5$.

19. Multiply $(a-b)^m$ by $2(a-b)^{-n}$. *Ans.* $2(a-b)^{m-n}$.

20. Multiply $3(x+y)^n$ by $ab(x+y)^n$.

21. Multiply $3(x+y)^n$ by $2(x+y)^{-m}$.

22. Multiply $-a(c+d)$ by $ab^2(c+d)$. *Ans.* $-a^2b^2(c+d)^2$.

23. Multiply $2a(a+b)^2$ by $-(a+b)^3$. *Ans.* $-2a\,(a+b)^5$.

24. Multiply $-2x^2(m-n)^3$ by $-x(m-n)^{-m}$.
Ans. $2x^3(m-n)^{3-m}$.

CASE II.

A Polynomial by a Monomial.

72.—Ex. 1. Multiply $c-d+b$ by a.

SOLUTION. Each term of the multiplicand must be taken a times. a times c is ac; a times $-d$ is $-ad$, and a times b is ab. Hence, $c-d+b$ multiplied by a is $ac-ad+ab$.

$$\begin{array}{r} c-\ d+\ b \\ \underline{a \qquad\quad} \\ ac-ad+ab \end{array}$$

2. What is the product of $2a+bc$ by a^2c?

3. What is the product of $4ac-3ax$ by $7ax$?

73. Rule for Multiplying a Polynomial by a Monomial.—*Multiply each term of the multiplicand by the multiplier.*

PROBLEMS.

1. Multiply $5ab+4c$ by $-3ac$. *Ans.* $-15a^2bc-12ac^2$.
2. Multiply $7a^2c-4ac^2$ by $7acx$.
3. Multiply $13axy+4x$ by $3a^2xy$. *Ans.* $39a^3x^2y^2+12a^2x^2y$.
4. Multiply $-11ay+4ax$ by $7axy$.
5. Multiply $2acx-3ay$ by $-4ax$. *Ans.* $-8a^2cx^2+12a^2xy$.
6. Multiply $3x^2-4y^2+5z^2$ by $2x^2y$.
7. Multiply $2xy^2z^3+3x^2y^3z-5x^2yz^2$ by $2xy^2z$.
8. Multiply $-3+ax-1+by$ by $-a$. *Ans.* $3a-a^2x+a-aby$.
9. Multiply $3mx-4ny$ by $-mn$.
10. Multiply x^m-y^n by $3x^2y$. *Ans.* $3x^{m+2}y-3x^2y^{n+1}$.
11. Multiply $a^2-2ab+b^2$ by $-3a^3b$.
12. Multiply $a^m+4a^{m-1}+6b^m$ by $3ab$. *Ans.* $3a^{m+1}b+12a^mb+18ab^{m+1}$.

CASE III.

A Polynomial by a Polynomial.

74.—Ex. 1. Multiply $5a-2b$ by $a+b$.

SOLUTION. $(a+b)$ times $(5a-2b)$ is a times $(5a-2b)$, plus b times $(5a-2b)$. a times $(5a-2b)$ is $5a^2-2ab$, and b times $(5a-2b)$ is $5ab-2b^2$. The sum of the partial products is $5a^2+3ab-2b^2$.

$$\begin{array}{r} 5a-2b \\ \underline{a+b} \\ 5a^2-2ab \\ \underline{5ab-2b^2} \\ 5a^2+3ab-2b^2 \end{array}$$

2. Multiply $2x+3y$ by $x+y$.

3. Multiply $x+y$ by $x-3y$

75. Rule for Multiplying a Polynomial by a Polynomial.—*Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

PROBLEMS.

(1.)

$$\begin{array}{l} x^m-y^n \\ \underline{x-y} \\ x^{m+1}-xy^n \\ \underline{\phantom{x^{m+1}-xy^n}-x^my+y^{n+1}} \\ x^{m+1}-xy^n-x^my+y^{n+1} \end{array}$$

(2.)

$$\begin{array}{l} ab-cd \\ \underline{ab+cd} \\ a^2b^2-abcd \\ \underline{+abcd-c^2d^2} \\ a^2b^2-c^2d^2 \end{array}$$

3. Multiply $a+b$ by $a-b$. *Ans.* a^2-b^2.

4. Multiply $a-b$ by $a-b$.

5. Multiply $2a+2b$ by $a+2c$. *Ans.* $2a^2+2ab+4ac+4bc$.

6. Multiply $2x-y$ by $2y+x$.

7. Multiply x^3+x^2+x-1 by $x-1$. *Ans.* x^4-2x+1.

8. Multiply x^2-3ax by $x+3a$.

9. Multiply a^2-ab+b^2 by $a+b$. *Ans.* a^3+b^3.

10. Multiply $1+4x-10x^2$ by $1-6x+3x^2$. *Ans.* $1-2x-31x^2+72x^3-30x^4$.

11. Multiply $2b^2+3ab-a^2$ by $7a-5b$.

12. Multiply $x+2m-1$ by $x+1$. *Ans.* $x^2+2mx+2m-1$.

13. Multiply x^2-xy+y^2 by $x+y$.

14. Multiply $x+2y-3z$ by $x-2y+3z$. *Ans.* $x^2-4y^2+12yz-9z^2$.

15. Multiply a^m+c^m by a^m-c^n.

16. Multiply x^m-y^m by x^m-y^m. *Ans.* $x^{2m}-2x^my^m+y^{2m}$.

17. Multiply $2a-b$ by $3a^2-1$. *Ans.* $6a^3-3a^2b-2a+b$

18. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$.

19. Multiply $2c+\frac{1}{2}$ by $2c+\frac{1}{2}$.

20. Multiply a^n+b^n by $a-b$. *Ans.* $a^{n+1}+ab^n-a^nb-b^{n+1}$.

76. The multiplication of polynomials may be *indicated* by enclosing each of the factors in a parenthesis and writing them one after the other; and the algebraic expression is said to be *expanded* when the multiplication thus indicated is performed.

1. Indicate the multiplication of $3ab^2-cd^3$ by xy^2+9. *Ans.* $(3ab^2-cd^3)(xy^2+9)$.

2. Expand $(2x^3-19x^2+26x-16)(x-8)$. *Ans.* $2x^4-35x^3+178x^2-224x+128$.

3. Expand $(y^5+1)(y+1)$.

4. Expand $(x^2+xy+y^2)(x-y)$. *Ans.* x^3-y^3.

5. Expand $(3ac-b)(6x-1)$.

6. Expand $(c^m+d^n)(c^m+d^n)$. *Ans.* $c^{2m}+2c^md^n+d^{2n}$.

7. Expand $(a+b+c)(a+b+c)$.

8. Expand $(x^2-x+1)(x^2+x+1)(x^4-x^2+1)$. *Ans.* x^8+x^4+1.

SECTION VII.

DIVISION.

77. Division is the process of finding how many times one quantity is contained in another; or,

Division is the process of separating a product into two factors, one of which is given.

The quantity to be divided or separated into two factors is called the **Dividend,** the quantity or factor given to divide by is called the **Divisor,** and the result of the division is called the **Quotient.**

The part of the dividend, if any, remaining undivided is called the **Remainder.**

EXERCISES.

78.—Ex. 1. How many times is $2a$ contained in $6a$?

SOLUTION.—$2a$ is contained in $6a$ 3 times, since the product of 3 by $2a$ is $6a$.

2. How many times is $5x$ contained in $15x$? In $25x$?

3. How many times is a contained in aby? In $2ab$?

4. What is the coefficient of the quotient of $6a$ divided by 2? Of $25x$ divided by $5x$?

5. If we divide ab by a, what is the quotient? If we divide ab by b, what is the quotient?

6. If we omit from the term xy the factor x, by what will that term have been divided?

7. If we omit from the term a^2 the factor a, what will be the result? If we omit from a^3 the factor a, what will be the result?

8. If we divide a^3 by a, what is the quotient? If we divide a^3 by a^2, what is the quotient?

9. If we divide a^5 by a^2, the quotient is a^3; what is the exponent of the factor in the quotient? What is the exponent of

the factor a in the dividend diminished by the exponent of the factor a in the divisor?

10. If we multiply $+3$ by $+2$, the product is $+6$; what then is the quotient of $+6$ divided by $+3$? by $+2$?

11. If we multiply -3 by -2, the product is $+6$; what then is the quotient of $+6$ divided by -3? by -2?

12. What is the quotient of $+10a$ divided by $-5a$?

13. If we multiply $+3$ by -2, the product is -6; what then is the quotient of -6 divided by $+3$?

14. What is the quotient of $-6a$ divided by $+3a$?

15. If we multiply -3 by $+2$, the product is -6; what is the quotient of -6 divided by -3?

16. What is the quotient of $-6a$ divided by $-3a$?

79. Principles.—1. *The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.*

Thus, $15ax \div 5a = 3x$, for $3x \times 5a = 15ax$.

2. *To omit a factor from a term is to divide by that factor.*

Thus, axy with a omitted, or xy, is the same as $axy \div a$, which is equal to xy; for $xy \times a = axy$.

3. *The exponent of a factor in the quotient is equal to its exponent in the dividend diminished by its exponent in the divisor.*

Thus, $a^5 \div a^2$, or a^5 with the factor a^2 omitted, is equal to a^3, for $a^3 \times a^2 = a^5$.

4. *The quotient is positive when the dividend and divisor have like signs, and negative when the dividend and divisor have unlike signs.*

Thus, $+ab$ divided by $b = +a$, for $+a \times (+b) = +ab$;
$-ab$ divided by $-b = \div a$, for $+a \times (-b) = -ab$;
$+ab$ divided by $-b = -a$, for $-a \times (-b) = +ab$;
$-ab$ divided by $+b = -a$, for $-a \times (+b) = -ab$.

CASE I.

A Monomial by a Monomial.

80.—Ex. 1. Divide $18ab$ by $6a$.

Solution. Dividing the dividend $18ab$ by 6 and a, the factors of the divisor, by rejecting those factors from the dividend, we have the quotient $3b$; for $3b$ is such a quantity as, multiplied by the divisor $6a$, will give the dividend $18ab$.

$$18ab \div 6a = 3b$$

2. Divide $8a^5b$ by $4a^2$.

Solution. Dividing the dividend $8a^5b$ by 4 and a^2, factors of the divisor, by rejecting those factors from the dividend, we have the quotient $2a^3b$.

$$8a^5b \div 4a^2 = 2a^3b$$

3. Divide $-6a^2b$ by $2a$. *Ans.* $-3ab$.

4. Divide $12a^5b^3$ by $-3a^2b$. *Ans.* $-4a^3b^2$.

81. Rule for Dividing one Monomial by another.—*Divide the numerical coefficient of the dividend by the coefficient of the divisor; annex to this quotient the letters of the dividend, giving each an exponent equal to its exponent in the dividend less the exponent in the divisor, and make the result positive when the dividend and divisor have like signs, and negative when they have unlike signs.*

When the dividend and divisor contain an equal literal factor, it is cancelled, and does not appear in the quotient.

PROBLEMS.

1. Divide $12abc$ by $3c$. *Ans.* $4ab$.

2. Divide $-12abc$ by $-4ab$.

3. Divide $24x^2y$ by $3xy$. *Ans.* $8x$.

4. Divide $-16x^2y^2z^2$ by $-4xz$. *Ans.* $4xy^2z$.

5. Divide $26ax^2y^2$ by $-2xy$.

6. Divide $24a^4b^5c^6$ by $-3a^2b^3c^4$. *Ans.* $-8a^2b^2c^2$.

7. Divide $20a^4b^4x^3y^3$ by $5b^2x^3y$.

8. Divide a^nb^3 by ab. *Ans.* $a^{n-1}b^2$.

9. Divide $-24a^2bn^3x^5$ by $-3an^3x^3$.

10. Divide $-b^3$ by b^2. *Ans.* $-b$.

11. Divide a^{m+n} by a^{2n}. *Ans.* a^{m-n}.

12. Divide $-12a^2b^3y^4z^3$ by $-aby^3z$.

13. Divide $a^{n+1}b^n$ by ab^2. *Ans.* a^nb^{n-2}.

14. Divide $-9(a+b)^2$ by $-3(a+b)$.

15. Divide $12(a^n-b^m)^2$ by $4(a^n-b^m)$.

16. Divide $(x-y)^5$ by $(x-y)^3$.

17. Divide $8a^{m-1}b^nc^3$ by $-2a^{m+1}b^2c^n$. *Ans.* $-4a^{-2}b^{n-2}c^{3-n}$.

CASE II.

A Polynomial by a Monomial.

82.—Ex. 1. Divide $6a^2b^2-4a^2b+2a^2c$ by $2a$.

$$2a\overline{)6ab^2-4a^2b+2a^2c}$$
$$3b^2-2ab+ac$$

SOLUTION. Dividing the first term of the dividend by $2a$, we have $3b^2$; dividing the second term by $2a$, we have $-2ab$; and dividing the third term by $2a$, we have ac. Uniting these results by their proper signs, we have, as the entire quotient, $3b^2-2ab+ac$.

2. Divide a^2+ac by a. *Ans.* $a+c$.

3. Divide $6a^4-3a^3x^2$ by $3a^2$. *Ans.* $2a^2-ax^2$.

83. Rule for Dividing a Polynomial by a Monomial.—*Divide each term of the dividend by the divisor, and connect the several results.*

PROBLEMS.

1. Divide $4x^3-8x^2+16x$ by $4x$. *Ans.* x^2-2x+4.
2. Divide $3a^4-12a^3+15a^2$ by $-3a^2$. *Ans.* $-a^2+4a-5$.
3. Divide $16a^2b-30a^3b$ by $4ab$.
4. Divide $4axy-4a+12ab$ by $4a$. *Ans.* $xy-1+3b$.
5. Divide $-14ax^3+7x^3$ by $-7x$.
6. Divide $10a^2x-15x^2$ by $5x$. *Ans.* $2a^2-3x$.
7. Divide $15a^2bc-10a^3b^4c^5y^2+5a^2b^3d^2$ by $-5abc$.
8. Divide $4a^2b-6ab^2$ by $-2ab$.
9. Divide ab^2+ac-a by a.
10. Divide $2a(x+y)^2+6a^2(x+y)$ by $2a$.
    Ans. $(x+y)^2+3a(x+y)$.
11. Divide $ax^2(c-d)-a^2x(c-d^2)$ by ax.
12. Divide $x^{n+1}-x^{n+2}+x^{n+3}-x^{n+4}$ by x^n.
    Ans. $x-x^2+x^3-x^4$.

CASE III.

A Polynomial by a Polynomial.

84.—Ex. 1. Divide x^3+2x^2-3x by x^2+3x.

$$\begin{array}{r|l} x^2+3x)\,x^3+2x^2-3x & (x-1 \\ \underline{x^3+3x^2} & \\ -x^2-3x & \\ \underline{-x^2-3x} & \end{array}$$

SOLUTION. The divisor and dividend for convenience are arranged according to the powers of x, beginning with the highest power. Since x^3, the first term of the dividend, must equal the product of x^2, the first term of the divisor, by the first term of the quotient, we divide x^3 by x^2, and have x for the first term of the quotient. x times the whole divisor is x^3+3x^2, which subtracted from the whole dividend leaves $-x^2-3x$.

Since $-x^2$, the first term of this new dividend, must equal the product of x^2, the first term of the divisor, by the second term of the quotient, we divide $-x^2$ by x^2, and have -1 for the second term of the quotient. -1 time the whole divisor is $-x^2-3x$, which subtracted from the last dividend leaves no remainder. Hence, the quotient is $x-1$.

2. Divide $a^2+2ab+b^2$ by $a+b$. *Ans.* $a+b$.

3. Divide $a^2-2ab+b^2$ by $a-b$. *Ans.* $a-b$.

85. Rule for Dividing a Polynomial by a Polynomial.—*Arrange the divisor and dividend according to the powers of one of their letters.*

Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient; multiply the whole divisor by it, and subtract the product from the dividend.

Consider the remainder as a new dividend, find the second term of the quotient in like manner as before, and thus continue till the first term of the divisor is not contained in the dividend.

If there be at last a remainder, write it, with the divisor under it, as a fractional part of the quotient.

PROBLEMS

1. Divide x^3-a^3 by $x-a$.

SOLUTION.

$$
\begin{array}{r|l}
x-a)\,x^3-a^3 & \underline{x^2+ax+a^2} \\
\underline{x^3-ax^2\quad} & \\
ax^2-a^3 & \\
\underline{ax^2-a^2x} & \\
a^2x-a^3 & \\
\underline{a^2x-a^3} &
\end{array}
$$

2. Divide a^2+x^2 by $a-x$.

SOLUTION.

$$
\begin{array}{l}
a-x)\,a^2+x^2\ \left(a+x+\dfrac{2x^2}{a-x}\right. \\
\underline{a^2-ax\quad} \\
\qquad ax+x^2 \\
\qquad \underline{ax-x^2} \\
\qquad\quad 2x^2
\end{array}
$$

3. Divide a^2-x^2 by $a-x$. *Ans.* $a+x$.

4. Divide x^2+6x+5 by $x+1$.

5. Divide $m^3-10m^2+27m-18$ by $m-6$.
Ans. m^2-4m+3.

6. Divide $6x^2+13xy+6y^2$ by $2x+3y$. *Ans.* $3x+2y$.

7. Divide $a^2+2ab+b^2$ by $a+b$.

8. Divide x^3-y^3 by $x-y$. *Ans.* x^2+xy+y^2.

9. Divide x^2-x-6 by $x-3$.

10. Divide $b-3b^2+3b^3-b^4$ by $b-1$. *Ans.* $-b+2b^2-b^3$.

11. Divide $3a^2-13ax+14x^2$ by $3a-7x$. *Ans.* $a-2x$.

12. Divide $x^2-7x+12$ by $x-3$.

13. Divide x^2+x-72 by $x+9$. *Ans.* $x-8$.

14. Divide $7x^3-24x^2+58x-21$ by $7x-3$.
Ans. x^2-3x+7.

15. Divide m^4+4m+3 by m^2+2m+1.

16. Divide x^4-13x^2+36 by x^2+5x+6.

17. Divide $a^2+2ab+b^2-c^2$ by $a+b-c$.

18. Divide x^4+y^4 by $x+u$. *Ans.* $x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}$.

Zero and Negative Exponents, and Reciprocals.

86. The **Reciprocal** of a quantity is the quotient arising from the division of 1 by that quantity.

Thus, the reciprocal of a is $\dfrac{1}{a}$, and the reciprocal of $a+b$ is $\dfrac{1}{a+b}$.

87. A **Zero Exponent,** or ^{0}, is used to preserve the trace of a letter, which might otherwise disappear in the process of division.

Thus, the quotient of x^3y^2 divided by x^3y may be written x^0y.

88. A **Negative Exponent** of a letter in a quotient indicates that the exponent of that letter in the divisor is greater than it is in the dividend.

Thus, the quotient of a^3 divided by a^5 is a^{3-5}, or a^{-2}.

89. Principles.—1. *Any quantity whose exponent is zero is equal to* 1.

For, $a^n \div a^n$ is $a^{n-n}=a^0$; also, $a^n \div a^n = 1$. Hence, $a^0=1$.

2. *Any quantity with a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.*

For, $a^3 \div a^5 = a^{-2}$; also, $a^3 \div a^5 = \frac{a^3}{a^5} = \frac{1}{a^2}$. Hence, $a^{-2} = \frac{1}{a^2}$, and

3. *Any factor may be transferred from the divisor to the dividend, or from the dividend to the divisor, if the sign of its exponent be changed from* + *to* −, *or from* − *to* +.

PROBLEMS.

1. Divide $-ab$ by ab. *Ans.* $-a^0b^0$, or -1.
2. Divide $8ab^2x$ by $2a^2b^3x^3$.
3. Express as a fraction the value of x^{-m}. *Ans.* $\frac{1}{x^m}$.
4. Express as a fraction the value of y^{-3}.
5. Express without a fraction the value of $\frac{1}{x^{m-n}}$. *Ans.* x^{n-m}.
6. Express without a negative exponent $\frac{11}{3}a^{-3}b^{-1}$.
7. Divide $-33a^{-2}b^{-3}x$ by $11a^3b^{-5}$. *Ans.* $-3a^{-5}b^2x$.
8. Divide $4a^4b^2$ by $2a^3b^3c^2$.
9. What is the reciprocal of $5a^2$? *Ans.* $\frac{1}{5a^2}$, or $5a^{-2}$.
10. Divide $25x^4y^3$ by $5x^5y^4$.

SECTION VIII.

REVIEW PROBLEMS.

90.—Ex. 1. What is the sum of $2a+3b-4c$, $-3a+4b-c$, $4a+7b+7c$, $a-b-4c$ and $-5a+2b-6c$? *Ans.* $-a+15b-8c$.

2. From $5x^2-2xy+3y^2$ subtract $-4x^2-2xy+7y^2$

3. From $2x^2-3xy+y$ take $4x^2+4xy-2y^2$.

4. What is the sum of $7-xy$, $ab+2$ and $3xy-8c-6$?

5. From $ax-bc+3ax+7bc$ take $4bc-2ax+bc+4ax$.

6. Multiply $c+1$ by $c+1$. *Ans.* c^2+2c+1.

7. Expand $(m-n)(m-n)$.

8. Divide $12x^4-192$ by $3x-6$.

9. Multiply $x^2+4xy+y^2$ by $x+y$. *Ans.* $x^3+5x^2y+5xy^2+y^3$.

10. Multiply $a^3+a^2b+ab^2+b^3$ by $a-b$.

11. Divide $20a^4-12a^3x$ by $4a^{-3}$.

12. Divide $a^2-b^2-2bc-c^2$ by $a-b-c$.

13. Simplify the expression $15a^2b^0+c^0d^3-5a^2+x$.

14. What is the reciprocal of x^2y^3z?

15. What is the product of $7x+3y-4z$ by $7x+4z$?

16. What is the value of $(1+2x+x^2)(x-1)$?

17. What is the value of $(a^3+b^3)\div(a^2-ab+b^2)$?
Ans. $a+b$.

18. What is the value of $(9axy+7a^3bc^2dx)(4a^3c^3d^2)$ divided by $2a^3b$? *Ans.* $18ab^{-1}c^3d^2xy+14a^3c^5d^3x$.

19. Divide $x^4+(2b^2-a^2)x+b^4$ by x^2+ax+b^2.

20. What is the value of $(a-2)(a-1)(a+1)(a+2)$ divided by a^2-a-2?

Test Questions.

91.—1. What is *Multiplication?* The multiplicand? The multiplier? The product? What are the factors of the product? What principles of multiplication are given? What is the rule for multiplying a monomial by a monomial? A polynomial by a monomial? A polynomial by a polynomial? How may the multiplication of polynomials be indicated?

2. What is *Division?* The dividend? The divisor? The quotient? A remainder? What principles of division are given? What is the rule for dividing a monomial by a monomial? A polynomial by a monomial? A polynomial by a polynomial?

3. What is the *Reciprocal* of a quantity? What is a zero exponent? A negative exponent? To what is any quantity equal whose exponent is zero? To what is any quantity with a negative exponent equal? How may any factor be transferred from the divisor to the dividend?

SECTION IX.

EQUATIONS.

92.—Ex. 1. By the use of what sign can you express the equality between two quantities?

2. What does $3x+6x=50-5$ express? What terms in the expression are on the left of the sign of equality? What terms are on the right of the sign of equality?

3. If the terms on each side of the sign of equality in the expression $3x+6x=50-5$ be united, what will the expression become?

4. If $9x$ be equal to 45, what is the value of x? If $3x+6x$ be equal to $50-5$, what is the value of x?

Definitions.

93. An **Equation** is an expression of the equality of two quantities.

The *first member* of an equation is the quantity on the left, and the *second member* is the quantity on the right, of the sign of equality.

Thus, $4x+10x=70$ is an equation of which $4x+10x$ is the first member, and 70 is the second member.

94. The **Transformation** of an equation is the process of changing its form without affecting the equality of the two members.

95. The **Solution** of an equation is the process of finding the value of the unknown quantity or quantities contained in the equation.

Transposition of Terms.

96.—Ex. 1. What is the value of $x+2-2$? What is the value of $5-2$?

2. If $x+2=5$, what will express the taking of 2 from each member of the equation? What will the equation become on uniting the terms of each member?

3. If 7 be taken from each member of the equation $x+7=15$, what will the equation become?

4. The sum of two numbers is 20, and the larger number is 4 more than the smaller. What is the smaller number?

5. What is the value of $x-5+5$? What is the value of $7+5$?

6. If $x-5=7$, what will express the adding of 5 to each member of the equation? What will the equation become on uniting the terms of each member?

7. John gave away 5 books, and had 13 left. How many had he at first?

8. Andrew has 2 less doves than Charles, and both together have 12 doves. How many doves has each?

WRITTEN EXERCISES.

97.—Ex. 1. Transpose $-c$ in the equation $x+a-c=b$ to the second member of the equation.

SOLUTION. Adding c to both members, which will not affect their equality (Ax. 1, Art. 24), we have $x+a=b+c$, in which $-c$ has been transposed with the sign changed.

$$\begin{array}{r} x+a-c=b \\ c=c \\ \hline x+a \quad =b+c \end{array}$$

2. Transpose c in the equation $x+a=b+c$ to the first member of the equation.

SOLUTION. Subtracting c from both members, which will not affect their equality (Ax. 2, Art. 24), we have $x+a-c=b$, in which c has been transposed with the sign changed.

$$\begin{array}{r} x+a \quad =b+c \\ c=c \\ \hline x+a-c=b \end{array}$$

3. Transpose cy in the equation $a+by=x-cy$ to the first member of the equation. *Ans.* $a+by+cy=x$.

4. Transpose ab in the equation $ab+cx=2ab$ to the second member of the equation. *Ans.* $cx=2ab-ab=ab$.

98. Rule for the Transposition of the Terms of an Equation. —*Any term may be transposed from one member of an equation to the other, if its sign be changed.*

PROBLEMS.

Transpose the terms of the following equations, so that all the terms containing the unknown quantity x shall be in the first member, and all the other terms in the second member.

1. $3d-2bc=ac-6x$. *Ans.* $6x=ac+2bc-3d$.

2. $5x+50=4x+56$. *Ans.* $5x-4x=56-50$.

3. $4x+12=2x-x+21$.

4. $5x-7a=2ab-ax$. *Ans.* $5x+ax=2ab+7a$.

5*

5. $48-2x=60-4x$.

6. $2x-4b=x+2$. *Ans.* $2x-x=4b+2$.

7. What is the value of x in the equation $2x+42=66-4x$?

SOLUTION. Transposing, we have $2x+4x=66-42$; uniting terms, we have $6x=24$, and dividing by the coefficient of the unknown term, which does not affect the equality of the members (Ax. 4, Art. 24), we have $x=4$.

$$2x+42=66-4x$$
$$2x+4x=66-42$$
$$6x=24$$
$$x=4$$

This value of x may be *verified* by substituting it in the given equation. Thus, $2\times4+42=66-4\times4$, or $8+42=66-16$.

8. Given $5x-4=2x+14$, to find x. *Ans.* $x=6$.

9. Given $2x-1=5x-7$, to find x.

10. Given $11x+16=9x+26$, to find x. *Ans.* $x=5$.

11. Given $5x+50=4x+56$, to find x.

12. Given $4x-8=-3x+13$, to find x. *Ans.* $x=3$.

13. Given $3ax-3ab=12c$, to find x. *Ans.* $x=b+\frac{4c}{a}$.

14. Given $4x-8=3x+20$, to find x.

15. Given $5x-15=2x+6$, to find x. *Ans.* $x=7$.

16. Given $40-6x-16=120-14x$, to find x.

17. Given $ax+a=ab$, to find x. *Ans.* $x=b-1$.

18. Given $x+55-2x=5x-11$, to find x.

19. Given $(x+2)3=(x-2)7$, to find x. *Ans.* $x=5$.

20. Given $nx-cx=nx-2x$, to find x.

21. Given $2x-2(30-x)=60-2x$, to find x. *Ans.* $x=20$.

22. Given $(2x+8)5=(32+x)3$, to find x.

23. Given $(6x-7)(2x-3)=(4x-5)(3x-4)$, to find x.

24. Given $12x-8-(8x-6)-(12-3x)=0$, to find x.

25. Given $(x-7)(x-3)-1=(x-9)(x-2)$, to find x.

SECTION X.

PROBLEMS PRODUCING EQUATIONS REQUIRING TRANSPOSITION.

99. The **Statement** of a problem consists in expressing in algebraic language the conditions of the problem.

An equation is thus obtained whose solution is the solution of the problem.

100.—Ex. 1. What are the two numbers whose sum is 42, and whose difference is 18?

SOLUTION. Let x denote the smaller number; then $x+18$ will denote the larger. By the conditions, $x+x+18=42$. Transposing and uniting terms, we have $2x=24$, and dividing, $x=12$ and $x+18=30$.

$$x = \textit{the smaller number;}$$
$$x+18 = \textit{the larger number.}$$
$$x+x+18=42$$
$$2x=24$$
$$x=12, \textit{ the smaller number;}$$
$$x+18=30, \textit{ the larger number.}$$

2. Divide 68 into two such parts that 84 diminished by the greater shall be equal to 3 times 40 diminished by the less.

SOLUTION. Let x denote the less part, then $68-x$ will denote the greater part, and by the conditions of the question $84-(68-x)=3(40-x)$, or $16+x=120-3x$. Transposing and uniting terms, we have $4x=104$; whence, $x=26$ and $68-x=42$.

$$x = \textit{the less part;}$$
$$68-x = \textit{the greater part.}$$
$$84-(68-x)=3(40-x)$$
$$16+x=120-3x$$
$$4x=104$$
$$x=26$$
$$68-x=42$$

3. Divide 100 into two such parts that 120 diminished by the greater is two times the less.

4. Three men engaged in a speculation requiring an outlay of \$8500. Of that amount A contributed a certain sum, B contributed as much as A and \$1000 more, and C contributed as much as both A and B, lacking \$1500. How much did each contribute? *Ans. A, \$2000; B, \$3000; C, \$3500.*

5. Divide 179 into two such parts that one part shall exceed twice the other by 17.

101. General Rule for Solving Problems Producing Equations. —*Denote one of the unknown quantities by x; and from the given conditions find an expression for each of the other unknown quantities, if any.*

Express in algebraic language the processes that would be necessary to verify the value of the unknown quantity or quantities, if they were already known.

Determine the value of the unknown quantity or quantities in the equations thus formed.

PROBLEMS.

1. A farmer has 400 sheep so distributed into two flocks, that the greater number increased by 25 is equal to twice the less diminished by 4 times 25. What is the number in each flock?

Ans. 225 in the greater; 175 in the less.

2. A has \$800, and B \$500; what sum must B give to A in order that A may have twice as much as B?

3. Two persons, A and B, speculate with equal sums of money; A gained \$126 and B lost \$87, and then A had twice as much money as B. What sum had each at first?

4. B is twice as old as A, and 22 years ago he was 3 times as old. What are their ages?

Ans. A's, 44 years; B's, 88 years.

5. A father is 3 times as old as his son; 4 years ago the father was 4 times as old as his son was. What is the age of each?

6. A is twice as old as B, and 7 years ago their united ages amounted to as many years as now represent the age of A. What are their ages?

7. A laborer engaged to work for 40 days, upon the conditions that for every day he worked he should receive \$2 and board, but for every day he was idle he was to pay \$.80. At the end of the time he received \$38. How many days did he work, and how many was he idle?

SOLUTION.

$$x = \textit{number of days he worked},$$
$$40 - x = \textit{number of days he was idle};$$
$$\$2x = \textit{sum earned},$$
$$\$.80(40 - x) = \textit{sum paid}.$$
$$\$2x - \$.80(40 - x) = \$38$$
$$\$2.80x = \$70$$
$$x = 25, \textit{ number of days he worked};$$
$$40 - x = 15, \textit{ number of days he was idle}.$$

8. A person hired a laborer to do a certain work on agreement that for every day he worked he should receive \$2, but for every day he was idle he should forfeit \$.50. He worked twice as many days as he was idle, and received \$42. How many days did he work?

9. A crew which can pull at the rate of 9 miles an hour on still water finds that it takes twice as long to go up a river as to go down. At what rate does the river flow?

Ans. 3 miles an hour.

10. From two towns, which are 187 miles apart, two travelers set out at the same time with the intention of meeting. One of them goes 8 miles and the other 9 miles a day. In how many days will they meet?

11. If \$56 should be added to the money I have, its amount would be trebled; how much money have I? *Ans. \$28.*

12. The sum of \$310 was raised by A, B and C together; B contributed \$30 more than A, and C \$40 more than B. How much did each contribute?

13. In a mixture of wine and water there are two gallons of wine for every three gallons of water. If a gallon of wine be

added, the mixture becomes half wine and half water. Of how many gallons did the mixture consist?

Ans. 5; 2 of wine and 3 of water.

NOTE. Let $2x$ = number of gallons of wine, $3x$ = the number of gallons of water, and $2x + 1$ = the number of gallons of wine after 1 gallon had been added.

14. After 34 gallons had been drawn out of one of two equal casks, and 80 gallons out of the other, there remained just 3 times as much in one cask as in the other. How much did each cask contain when full? *Ans. 103 gallons.*

15. After A had received $10 from B, he had as much money as B and $6 more, and together they have $40. How much had each at first?

16. A courier who travels 50 miles a day had been gone 5 days when a second was sent to overtake him, in order to do which he travels 75 miles a day. In what time will the second overtake the first?

17. A garrison had such a quantity of bread as would last 6 weeks, if distributed to each man at the rate of 10 ounces a day. Having immediately discharged 1200 men, the commander found that he had bread enough to last 8 weeks, allowing each man 12 ounces per day. What was the number of men at first in the garrison?

18. A has $600 and B $460; if A increases his capital by $4 per month, and B increases his by $1 per month, in how many months will A have twice as much money as B?

Ans. 160.

Test Questions.

102.—1. What is an *Equation?* Which is the first member of an equation? The second member?

2. What is the *Transformation* of an equation? The solution of an equation? The rule for the transposition of the terms of an equation?

3. Of what does the *Statement* of a problem consist? What is the general rule for solving problems producing equations?

SECTION XI.

FACTORS.

103.—1. Of what two integers is ab the product?

2. What two integers multiplied together produce xy? What three produce abc?

3. What integers are factors of bcd? Of $2x$?

4. Of what integers other than 1 is a^2x the product?

5. What common factors other than 1 have $3ac^2d$ and a^2cx?

6. Give the sets of integral factors greater than 1 which when multiplied together will produce $4a^2b^4$.

7. Of what quantities are a^2 and d the factors? 2, x and y, the factors?

8. In a^2 how many times is a a factor? In $16x^3$ how many times is 2 used as a factor? How many times is x used as a factor?

Definitions.

104. The **Factors** of a quantity are the integers which, being multiplied together, will produce that quantity.

105. A **Prime Quantity** is an integer that has no integral factors except itself and 1.

Thus, 17, 23 and a are prime quantities.

106. **Quantities** are prime to each other when they have no common factors except 1.

Thus, 17 and 20, and a and b are prime to each other.

107. A **Composite Quantity** is an integer that has other factors besides itself and 1.

Thus, 12 and ab are composite quantities.

108. The **Square** of a quantity is the product obtained by taking the quantity twice as a factor.

Thus, 4 is the square of 2; a^2, the square of a.

109. The **Square Root** of a quantity is one of the two equal factors of that quantity.

Thus, a is the square root of a^2, and $3ab$ is the square root of $9a^2b^2$.

110. One quantity is *divisible* by another when the latter is a factor of the former.

Thus, a^2b is divisible by a, a^2 or b.

COMPOSITION.

111. A **Theorem** is something to be proved or demonstrated.

112. **Composition** in algebra is the process of producing composite quantities.

Composite quantities may be produced by actual multiplication, and, in some instances, by abridged methods derived from the following theorems:

Theorem I.

113. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

For, let a and b represent two quantities, then $a+b$ will denote their sum, and $(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2$.

PROBLEMS.

1. Find the square of $m+n$. *Ans.* $m^2+2mn+n^2$.

2. Find the square of $a+c$.

3. What is the square of $x+2y$? *Ans.* $x^2+4xy+4y^2$.

4. What is the square of $c+3$?

5. What is the square of $3+a^2$? *Ans.* $9+6a^2+a^4$.

6. What is the square of $3a+2b$?

Theorem II.

114. *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

For, let a and b represent the two quantities, then $a-b$ will denote their difference, and $(a-b)^2=(a-b)(a-b)=a^2-2ab+b^2$.

PROBLEMS.

1. Find the square of $x-y$. *Ans.* $x^2-2xy+y^2$.

2. Find the square of $m-n$.

3. What is the square of a^2-1? *Ans.* a^4-2a^2+1.

4. What is the square of $3a^4-8a^2$?

5. Expand $(5x^3-2cd)(5x^3-2cd)$.

Theorem III.

115. *The product of the sum and difference of two quantities is equal to the difference of their squares.*

For, let a and b represent the two quantities, then $a+b$ will denote their sum, and $a-b$ their difference, and $(a+b)(a-b)=a^2-b^2$.

PROBLEMS.

1. Find the product of $(x+y)$ by $(x-y)$. *Ans.* x^2-y^2.

2. Find the product of $(m+n)$ by $(m-n)$.

3. What is the product of $(2a+b)$ by $(2a-b)$?
Ans. $4a^2-b^2$.

4. What is the product of (n^4+1) by (n^4-1)?

5. Expand $(2a-4b)(2a+4b)$.

6. Expand $(7x^2+4y)(7x^2-4y)$.

FACTORING.

116. Factoring is the process of finding the factors of a composite quantity.

Composite quantities may be factored by actual division, and, in some instances, by abridged methods derived from the preceding theorems (Art. 113–115), and, also, by the following theorems.

Theorem I.

117. *The difference of any two equal even powers of two quantities is divisible by the sum of the quantities.*

For, let a and b represent two quantities, a being greater than b; then, by actual division,

$$(a^2-b^2) \div (a+b) = a-b,$$
$$(a^4-b^4) \div (a+b) = a^3-a^2b+ab^2-b^3,$$
$$(a^6-b^6) \div (a+b) = a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5, \text{ and so on.}$$

Theorem II.

118. *The difference of any two equal powers of two quantities is divisible by the difference of the quantities.*

For, let a and b represent two quantities, a being greater than b; then, by actual division,

$$(a^2-b^2) \div (a-b) = a+b,$$
$$(a^3-b^3) \div (a-b) = a^2+ab+b^2,$$
$$(a^4-b^4) \div (a-b) = a^3+a^2b+ab^2+b^3, \text{ and so on.}$$

Theorem III.

119. *The sum of any two equal odd powers of two quantities is divisible by the sum of the quantities.*

For, let a and b represent two quantities, a being greater than b; then, by actual division,

$$(a^3+b^3) \div (a+b) = a^2-ab+b^2,$$
$$(a^5+b^5) \div (a+b) = a^4-a^3b+a^2b^2-ab^3+b^4,$$
$$(a^7+b^7) \div (a+b) = a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6, \text{ and so on.}$$

CASE I.

A Monomial into its Prime Factors.

120.—Ex. 1. Find the prime factors of $10a^2b^3$.

SOLUTION. The factors of 10 are 2 and 5; of a^2, a and a; and of b^3, b, b and b. Hence, the prime factors of $10a^2b^3$ are $2\times5\times a\times a\times b\times b\times b$.

$$\begin{aligned} 10&=2\times5\\ a^2&=a\times a\\ b^3&=b\times b\times b\\ \hline 10a^2b^3&=2\times5\times a\times a\times b\times b\times b\end{aligned}$$

2. Find the prime factors of $6a^2b^2c^3$.

3. Find the prime factors of $21x^2y^3z$.

121. Rule for Resolving a Monomial into its Prime Factors.—*Find the prime factors of the numerical coefficient, and after them write each letter as many times as there are units in its exponent.*

PROBLEMS.

1. Find the prime factors of $25ab^2c^3$.
Ans. $5, 5, a, b, b, c, c, c$.

2. Find the prime factors of $16x^2y^3$.

3. Find the prime factors of $21ab^{2n}c^{3n}$.
Ans. $3, 7, a, b^n, b^n, c^n, c^n, c^n$.

4. Resolve $42a^2b^{n+1}c^{m-1}$ into its prime factors.
Ans. $2, 3, 7, a, a, b^n, b, c^m, c^{-1}$.

CASE II.

A Polynomial into two Factors, a Monomial and a Polynomial.

122.—Ex. 1. Resolve $12ab^2+6ab-9ac$ into two factors.

SOLUTION. $3a$ is a factor of each term, and therefore a monomial factor of the polynomial. Dividing $12ab^2+6ab-9ac$ by $3a$, we have, for the other factor, the polynomial $4b^2+2b-3c$.

$$(12ab^2+6ab-9ac)\div3a=4b^2+2b-3c$$
$$12ab^2+6ab-9ac=3a(4b^2+2b-3c)$$

2. Find the two factors of ab^2+ad.

3. Resolve into two factors $4ax^2-6bx^2$.

123. Rule for Resolving a Polynomial into two Factors, a Monomial and a Polynomial.—*Divide the polynomial by the greatest monomial factor common to all the terms, and prefix the divisor, as a coefficient, to the quotient enclosed in a parenthesis.*

PROBLEMS.

1. Find the factors of $x^2+ax-bx$. *Ans.* $x(x+a-b)$.

2. Find the factors of $a-a^2b$.

3. Find the factors of $2xy+4xy^2-6x^2y$.
Ans. $2xy(1+2y-3x)$.

4. Resolve into factors ac^2+bc-d^2c.

5. Resolve into factors $7x^2y+28xy^2z-84x^2y^2$.

6. Resolve into factors $16m^2nx^ny+8m^2nx^ny^2-48m^2nx^n$.
Ans. $8m^2nx^n(2y+y^2+6)$.

CASE III.

A Trinomial into two Equal Binomial Factors.

124. A **Trinomial** can be resolved into two equal binomial factors when two of the terms are squares and positive, and the other term is twice the product of their square roots.

125.—Ex. 1. Find the factors of $a^2+2ab+b^2$.

SOLUTION. a^2 is the square of a, b^2 is the square of b, and $2ab$ is twice the product of a and b. Hence, the trinomial is, by Art. 113, the square of $(a+b)$, and has for its two equal factors $(a+b)$ and $(a+b)$.

$$a^2+2ab+b^2=(a+b)(a+b)$$

2. Find the factors of $a^2-2ab+b^2$.

SOLUTION. a^2 is the square of a, b^2 is the square of b, and $-2ab$ is minus twice the product of a and b. Hence, the trinomial is, by Art. 114, the square of $(a-b)$, and has for its two equal factors $(a-b)$ and $(a-b)$.

$$a^2-2ab+b^2=(a-b)(a-b)$$

3. Find the factors of $x^2-2xy+y^2$.

4. Find the factors of $c^2+2cd+d^2$.

126. Rule for Resolving a Trinomial into two equal Binomial Factors.—*Find the square roots of the square terms, and if twice the product of these roots equals the other term, these roots connected by the sign of the other term will be one of the two equal factors.*

PROBLEMS.

1. Find the factors of $a^2+4ax+4x^2$.
   Ans. $(a+2x)$ *and* $(a+2x)$.

2. Find the factors of $25x^2-10x+1$.

3. Find the factors of c^2-2c+1. *Ans.* $(c-1)$ *and* $(c-1)$.

4. Factor $1+4xy+4x^2y^2$.

5. Factor $1-2a^2+a^4$. *Ans.* $(1-a^2)$ *and* $(1-a^2)$.

6. Resolve $9a^2b^2+24a^2bd+16a^2d^2$.
   Ans. $(3ab+4ad)$ *and* $(3ab+4ad)$.

CASE IV.

A Binomial into two Binomial Factors.

127. A **Binomial** can be resolved into two binomial factors, when the binomial is the difference between two squares.

128.—Ex. 1. Find the factors of a^2-b^2.

SOLUTION. a^2 is the square of a, and b^2 is the square of b; hence, by Art. 115, the factors of a^2-b^2 must be the sum and difference of a and b, or $(a+b)$ and $(a-b)$.

$$a^2-b^2=(a+b)(a-b)$$

2. Find the factors of a^2-d^2. *Ans.* $(a+d)$ *and* $(a-d)$.

3. Find the factors of $9x^2-4y^2$.
Ans. $(3x+2y)$ *and* $(3x-2y)$.

129. Rule for Resolving a Binomial into two Binomial Factors. —*Find the square root of each term, and make their sum one factor and their difference the other.*

PROBLEMS.

1. Find the factors of $4a^2-b^2$. *Ans.* $(2a+b)(2a-b)$.

2. Find the factors of n^2-1.

3. Find the factors of $25-b^2$. *Ans.* $(5+b)(5-b)$.

4. Resolve $49x^4-16y^2$ into two factors.

5. Resolve $a^{2m}-b^{2n}$ into two factors.

6. Resolve $a^8b^4-x^4y^2$ into two factors.

130. Any **Binomial** consisting of the difference of the same powers of two quantities, or the sum of the same odd powers, can be factored by aid of Articles 117 and 119. For the quotient and divisor must be factors of the dividend.

7. Resolve a^3-x^3 into two factors.

SOLUTION. a^3-x^3 is divisible by $a-x$, by Art. 118; hence, $a-x$ is a factor of a^3-x^3. Dividing, we have, as the other factor, a^2+ax+x^2.

$$(a^3-x^3)\div(a-x)$$
$$=a^2+ax+x^2$$

8. Resolve a^3-1 into two factors. *Ans.* $(a-1)(a^2+a+1)$.

9. Resolve x^3+y^3 into two factors.

10. Resolve a^3+27b^3 into two factors.

11. Resolve a^4-c^4 into two factors.
Ans. $(a+c)(a^3-a^2c+ac^2-c^3)$, or $(a-c)(a^3+a^2c+ac^2+c^3)$.

12. Resolve $m^4 - n^4$ into its prime factors.

SOLUTION.

$$\begin{aligned} m^4 - n^4 &= (m^2+n^2)(m^2-n^2) \\ &= (m^2+n^2)(m+n)(m-n) \end{aligned}$$

13. Resolve $1 - x^4$ into its prime factors.

Ans. $(1+x^2)(1+x)(1-x)$.

14. Resolve $a^6 - 1$ into its four prime factors.

15. Resolve $a^6 - b^6$ into its prime factors.

Ans. $(a^2 - ab + b^2)(a^2 + ab + b^2)(a+b)(a-b)$.

SECTION XII.

DIVISORS.

131.—Ex. 1. What quantities will divide $2a$ without a remainder?

2. What quantities are factors of $2a$? Of $3ab$?

3. What factors are common to $6a^2b$ and $2ad$? What is the largest factor common to $6a^2b$ and $2ad$?

4. Why will no integral quantity greater than 1 divide both $5axy^2$ and $3bc^2d$ without a remainder?

Definitions.

132. A **Divisor** of a quantity is any factor of that quantity.

Thus, 2 and a are divisors of $2a$.

133. A **Common Divisor** of two or more quantities is any factor common to those quantities.

Thus, b is a common divisor of $5abc$ and $6bd$.

134. The **Greatest Common Divisor** of two or more quantities is the largest factor common to those quantities.

Thus, $4ac$ is the greatest common divisor of $4acy$ and $16acx^2$.

135. Quantities are prime to each other when they have no common factor.

Thus, abc and $3xy$ are prime to each other.

136. Principles.—1. *The greatest common divisor of two or more quantities is the product of all their common prime factors.*

Thus, all the common prime factors of $4acy$ and $16acx^2$ are 2, 2, a and c, and their product, $4ac$, is the greatest common divisor of those quantities.

2. *A divisor of a quantity is a divisor of any number of times that quantity.*

Thus, d, which is a divisor of bd^2, is also a divisor of a times bd^2, or abd^2.

3. *A common divisor of two quantities is a common divisor of their sum and of their difference.*

For, let ad and bd be two quantities whose common divisor is d; then their sum, $ad+bd$, or $(a+b)d$, and their difference, $ad-bd$, or $(a-b)d$, also have d as a common divisor.

4. *The common divisor of two quantities is not changed by cancelling from either quantity, or introducing into either, a factor not contained in the other quantity.*

For, let ac^2d and bdn be two quantities whose common divisor is d. Cancel a in the one, or introduce m into the other, and the common divisor will remain the same.

CASE I.

Greatest Common Divisor of Monomials.

137.—Ex. 1. Find the greatest common divisor of $9a^2b^3c$ and $21a^2b^2d$.

$$9a^2b^3c = 3\times 3\times a^2\times b^2\times bc$$
$$21a^2b^2d = 7\times 3\times a^2\times b^2\times d$$
$$\textit{Greatest com. div.} = 3\times a^2\times b^2 = 3a^2b^2$$

SOLUTION. Resolving the quantities into factors, we find that 3, a^2 and b^2 are all of the common factors. Hence, by Prin. 1, Art. 136, the greatest common divisor is the product of 3, a^2 and b^2, or $3a^2b^2$.

2. Find the greatest common divisor of $6a^3xy$ and $18a^2by^2$. *Ans.* $6a^2y$.

3. Find the greatest common divisor of $15x^5y^2$, $18ax^4y$ and $12bx^5y^2$. *Ans.* $3x^4y$.

138. Rule for Finding the Greatest Common Divisor of Monomials.—*Resolve the quantities into their prime factors, and find the product of all the common factors.*

PROBLEMS.

1. Find the greatest common divisor of $12a^2xy$ and $16acx^2y$. *Ans.* $4axy$.

2. Find the greatest common divisor of $10a^2bc$, $6ab^2c^2d$ and $4a^3b^3c$.

3. Find the greatest common divisor of $-24ac^2dx^3y$ and $28abcx^2y^2$. *Ans.* $4acx^2y$.

4. Find the greatest common divisor of $4a^{2n}b^2x^5$, $6a^nb^3x^2y$ and $8a^{3n}bm^2x$.

5. Find the greatest common divisor of $11a^3bc^2n^2$, $7b^3c^2m^2x^n$ and $19a^3b^2c^3n^2$.

CASE II.

Greatest Common Divisor of Polynomials.

139.—Ex. 1. What is the greatest common divisor of $x^2+8x+15$ and $x^2+9x+20$?

SOLUTION. Since any quantity is the greatest divisor of itself, if $x^2+8x+15$ be a divisor of $x^2+9x+20$, it must be the greatest common divisor required. We find it is not a divisor of $x^2+9x+20$, since, on trial, $x+5$ remains.

$$
\begin{array}{l}
x^2+8x+15)\,x^2+9x+20\,(1 \\
\qquad\qquad\quad \underline{x^2+8x+15} \\
\qquad\qquad\qquad\quad x+5)\,x^2+8x+15\,(x+3 \\
\qquad\qquad\qquad\qquad\qquad \underline{x^2+5x\quad\;} \\
\qquad\qquad\qquad\qquad\qquad\quad 3x+15 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{3x+15}
\end{array}
$$

If $x+5$ be a divisor of $x^2+8x+15$, it must also be a divisor of $x^2+9x+20$, which is once $x^2+8x+15$, plus $x+5$. On trial it is found to be a divisor of $x^2+8x+15$. Hence, $x+5$ is the greatest common divisor of the given quantities.

2. What is the greatest common divisor of $2x^2-7x+5$ and $3x^2-7x+4$?

$$\begin{array}{r}
3x^2-\ 7x+4 \\
2 \\
\hline
2x^2-7x+5\big)6x^2-14x+8\big(3 \\
6x^2-21x+15 \\
\hline
7\big)7x-7 \\
\hline
x-1\big)2x^2-7x+5\big(2x-5 \\
2x^2-2x \\
\hline
-5x+5 \\
-5x+5 \\
\hline
\end{array}$$

Solution. If we divide $3x^2-7x+4$ by $2x^2-7x+5$, the quotient of $3x^2$ divided by $2x^2$ is a fraction. To avoid this, we multiply the dividend by the factor 2, which will not change the common divisor (Prin. 4, Art. 136), and then divide.

If we now divide $2x^2-7x+5$ by $7x-7$, the first term of the quotient will be fractional. We therefore cancel the factor 7 in the new divisor, which does not change the common divisor (Prin. 4, Art. 136), and dividing by the remaining factor, $x-1$, find it to be the greatest common divisor required.

3. What is the greatest common divisor of x^3+x^2-2x and $x^4+2x^3+x^2+2x$? *Ans.* x^2+2x.

4. What is the greatest common divisor of $x^2-2x-80$ and $x^2+2x-120$? *Ans.* $x-10$.

140. Rule for Finding the Greatest Common Divisor of Polynomials.—*Divide the greater quantity by the less, and if there be a remainder, divide the divisor by it, and so continue to divide till there is no remainder. The last divisor will be the greatest common divisor.*

When there are more than two quantities, find the greatest common divisor of two of them, and then of that divisor and a third quantity, and so on for all the quantities.

In the process, a factor may be introduced into the dividend to make it divisible, or a factor may be cancelled from either divisor or dividend, which is not contained in the other.

The signs of either divisor or dividend, or both, may be changed without changing the common divisor.

PROBLEMS.

1. Find the greatest common divisor of a^2+ax and $a^2+2ax+x^2$. *Ans.* $a+x$.

2. Find the greatest common divisor of a^2-ax and $a^2-2ax+x^2$.

3. Find the greatest common divisor of $3x^2+x-2$ and $3x^2+4x-4$. *Ans.* $3x-2$.

4. Find the greatest common divisor of $x^2-7x+10$ and $4x^3-25x^2+20x+25$.

5. Find the greatest common divisor of $x^2-9x-36$ and $x^2-15x+36$.

SOLUTION.

$$x^2-9x-36)x^2-15x+36(1$$
$$\underline{x^2-\ \ 9x-36}$$
$$-\ 6x+72$$

Cancelling the factor 6 and changing the signs, $\Big\}$ $x-12)x^2-9x-36(x+3$

$$\underline{x^2-12x}$$
$$3x-36$$
$$3x-36$$

6. Find the greatest common divisor of $x^3-9x^2+23x-12$ and $x^3-10x^2+28x-15$. *Ans.* x^2-5x+3.

7. Find the greatest common divisor of $3x^3-24x-9$ and $2x^3-16x-6$.

8. Find the greatest common divisor of x^3+x-10 and x^4-16.

9. Find the greatest common divisor of a^2-4b^2, $a^2+ab-2b^2$ and $2a^2+3ab-2b^2$.

10. Find the greatest common divisor of x^3-6 and x^3-8.

11. Find the greatest common divisor of x^2+x-6 and $3x^2+6x-24$.

SECTION XIII.

MULTIPLES.

141.—Ex. 1. What quantity is a times $3b$? c times xy?

2. What quantity contains an integral number of times 2, 3, x and y?

3. What is the least quantity that will contain $4a$ and $7b$ an integral number of times?

4. What are the prime factors of $4a$ and $7b$? What is the product of all the prime factors of $4a$ and $7b$?

Definitions.

142. A **Multiple** of a quantity is any integral number of times that quantity.

Thus, $3a^2b^2$ is a multiple of ab.

143. A **Common Multiple** of two or more quantities is any quantity which is an integral number of times each of them.

Thus, $24a^2b^2$ is a common multiple of $3ab$ and $4ab$.

144. The **Least Common Multiple** of two or more quantities is the least quantity which is an integral number of times each of them.

Thus, $12ab$ is the least common multiple of $3ab$ and $4ab$.

145. Principles.—1. *A multiple of a quantity contains all the prime factors of that quantity.*

Thus, $3a^2b^2$, a multiple of ab, must contain the prime factors of ab.

2. *A common multiple of two or more quantities contains all the prime factors of those quantities.*

Thus, $24a^2b^2$, a common multiple of $3ab$ and $4ab$, must contain the prime factors of those quantities.

3. *The least common multiple of two or more quantities is the least quantity that contains all the prime factors of those quantities.*

Thus, $12ab$, the least common multiple of $3ab$ and ab, is the least quantity that contains all the prime factors of those quantities.

4. *The least common multiple of two or more quantities is equal to the quotient arising from dividing the product of the quantities by their greatest common divisor.*

For, the greatest common divisor of two or more quantities contains all the factors common to those quantities, and the least common multiple of the quantities must contain only the prime factors of the quantities.

CASE I.

Least Common Multiple of Monomials.

146.—Ex. 1. Find the least common multiple of $2a^2b^2c^3x$ and $4a^2b^3c^2y$.

$$\begin{aligned} 2a^2b^2c^3x &= 2 \times a^2 \times b^2 \times c^3 \times x \\ 4a^2b^3c^2y &= 2^2 \times a^2 \times b^3 \times c^2 \times y \\ \textit{Least com. mult.} &= 2^2 \times a^2 \times b^3 \times c^3 \times x \times y \\ &= 4a^2b^3c^3xy \end{aligned}$$

SOLUTION. Resolving the quantities into factors, we find that all the different factors of the two quantities are 2^2, a^2, b^3, c^3, x and y. Hence, the least common multiple is their product, or $4a^2b^3c^3xy$.

2. Find the least common multiple of $16a^4bc$ and $20a^3b^3d$.
Ans. $80a^4b^3cd$.

3. Find the least common multiple of $12a^3b^2c$ and $18ab^2c^3$.
Ans. $36a^3b^2c^3$.

147. Rule for Finding the Least Common Multiple of Monomials.—*Resolve the quantities into their prime factors, and find the product of all the different factors, each factor being taken the greatest number of times it occurs in any of the quantities.*

PROBLEMS.

1. Find the least common multiple of $8a^2x^2y^3$ and $12b^2x^3y^2$.
Ans. $24a^2b^2x^3y^3$.

2. Find the least common multiple of $5a^2b^2c^3$, $6a^3b^4c^3$ and $30a^4b^3c^2$.

3. Find the least common multiple of $3ab^3x^2y$, $7a^2b^2x^3y^4$ and $12a^3b^4x^2y^2$.
Ans. $84a^3b^4x^3y^4$.

4. Find the least common multiple of $15a^2b^2$, $12ab^3$ and $6a^3b$.

5. Find the least common multiple of $16m^2n^2p^3$, $12m^4n^3p^2$ and $24m^3n^2p^4$.
Ans. $48m^4n^3p^4$.

CASE II.

Least Common Multiple of Polynomials.

148.—Ex. 1. Find the least common multiple of x^2-a^2 and x^3-a^3.

SOLUTION. The greatest common divisor of x^2-a^2 and x^3-a^3 is $x-a$. Hence, by Prin. 4, Art. 145, the least common multiple of the quantities is $(x^3-a^3)(x^2-a^2)$, divided by $x-a$, or $x^4-a^3x+ax^3-a^4$.

$$\frac{(x^3-a^3)(x^2-a^2)}{x-a}=x^4-a^3x+ax^3-a^4$$

2. Find the least common multiple of $3a^2-4ax$ and $6a^3-8a^2x$.
Ans. $6a^3-8a^2x$.

149. Rule for Finding the Least Common Multiple of Polynomials.—*Divide the product of the quantities by their greatest common divisor.*

PROBLEMS.

1. Find the least common multiple of a^4-x^4 and $a^3-a^2x-ax^2+x^3$.
Ans. $a^5-a^4x-ax^4+x^5$.

2. Find the least common multiple of x^2-1, x^3+1 and x^3-1.

3. Find the least common multiple of $2x-1$ and $4x^2-1$.
Ans. $4x^2-1$.

4. What is the least common multiple of $x+1$ and x^2+2x+1?

SECTION XIV.

REVIEW PROBLEMS.

150.—Ex. 1. Find the square of $a+5$. *Ans.* $a^2+10a+25$.

2. Find the square of x^2-1.

3. Expand $(5a^2+2y)(5x^2-2y)$. *Ans.* $25x^4-4y^2$.

4. Resolve $12ab^2c^3mx^2$ into its prime factors.

5. Resolve $a^2+4ab+4b^2$ into factors. *Ans.* $(a+2b), (a+2b)$.

6. Resolve $16a^2-25b^2$ into two factors.

7. Resolve $1-8c^3$ into two factors.

8. Resolve x^6-y^6 into its prime factors.

9. Find a monomial factor of $x^3-x^2y+x^2y^5$. *Ans.* x^2.

10. Find, by factoring, the greatest common divisor of a^2-b^2 and $a^2-2ab+b^2$.

SOLUTION.

$$\begin{aligned} a^2-b^2 &= (a+b)(a-b) \\ a^2-2ab+b^2 &= (a-b)(a-b) \\ \textit{Greatest com. div.} &= a-b. \end{aligned}$$

11. Find, by factoring, the greatest common divisor of c^3-d^3 and c^2-d^2. *Ans.* $c-d$.

12. Find, by factoring, the least common multiple of a^2-b^2 and $a^2-2ab+b^2$.

13. Find the least common multiple of $a-1$, a^2-1, $a-2$ and a^2-4.

14. What is the least common multiple of $2(a+b)$ and $3(a^2-b^2)$?

15. Find the greatest common divisor and the least common multiple of $5a^2bc-10ab^2c$ and $3a^2bcd-6b^3cd$.

Test Questions.

151.—1. What are the *Factors* of a quantity? What is a prime quantity? When are quantities prime to each other? What is a composite quantity?

2. What is the *Square* of a quantity? The square root of a quantity? When is one quantity divisible by another?

3. What is *Composition* in algebra? To what is the square of two quantities equal? The square of the difference of two quantities? The product of the sum and difference of two quantities?

4. What is *Factoring?* By what is the difference of two equal even powers divisible? The difference of any two equal powers of two quantities? The sum of any two odd powers of two quantities?

5. What is the *Rule* for resolving a monomial into its prime factors? A polynomial into two factors, a monomial and a polynomial? A trinomial into two equal binomial factors? A binomial into two binomial factors?

6. What is a *Divisor* of a quantity? A common divisor? The greatest common divisor? When are quantities prime to each other? What are the given principles of divisors? What is the rule for finding the greatest common divisor of monomials? For finding the greatest common divisor of polynomials?

7. What is a *Multiple* of a quantity? A common multiple of two or more quantities? The least common multiple of two or more quantities? What are the given principles of multiples? What is the rule for finding the least common multiple of monomials? For finding the least common multiple of polynomials?

SECTION XV.

FRACTIONS.

152.—Ex. 1. If x represent the whole of anything, what part of the thing will $\frac{x}{2}$ represent?

2. If a whole orange be represented by x, what part of it will $\frac{x}{3}$ represent? What part of it will $\frac{2x}{3}$ represent?

3. If one-fourth of x is represented by $\frac{x}{4}$, how many fourths of x are represented by $\frac{5x}{4}$?

4. If x represents a quantity, how many times that quantity is represented by $2x$? By $2x+\frac{x}{2}$?

5. If the sign $-$ written before x shows that the quantity x is to be subtracted, what does $-\frac{x}{2}$ denote?

6. When x is equal to 6, what is the value expressed by $\frac{x}{2}$? By $\frac{x}{3}$? By $x+\frac{x}{2}$?

Definitions.

153. A **Fraction** is one or more of the equal parts into which a unit is divided.

154. The **Denominator** of a fraction is the quantity which numbers the parts of the unit, and the **Numerator** is the quantity which numbers the parts taken.

155. The **Terms** of a fraction are the denominator and numerator, and are written by placing the numerator above the denominator, with a line between them.

Thus, the fraction $\frac{a}{b}$ denotes that a unit has been divided into b equal parts, and that a of the parts are taken.

156. An **Integer,** or entire quantity, is one which has no fractional part.

Thus, ab and $x+y$ are integers.

157. A **Mixed Quantity** is one having an integer and a fractional part.

Thus, $a+\frac{b}{d}$ is a mixed quantity.

158. The **Sign** of a fraction is the sign written before the dividing line, to show whether the fraction is to be added or subtracted.

Thus, in $-\frac{a}{b}$ the sign of the fraction is $-$, and shows that the fraction is to be subtracted.

159. Principles.—1. *The value of a fraction is the quotient obtained by dividing the numerator by the denominator.*

For, any fraction may be regarded as an expression of division. Thus, let $\frac{ab}{b}$ be any fraction, and its value must be $ab \div b$, or a.

2. *Multiplying the numerator or dividing the denominator of a fraction by any quantity, multiplies the fraction by that quantity.*

For, let $\frac{ab}{b}$ be any fraction and b any integer. Then, multiplying the numerator by b, we have

$$\frac{ab \times b}{b} = \frac{ab^2}{b} = ab,$$

and dividing the denominator by b, we have

$$\frac{ab}{b \div b} = \frac{ab}{1} = ab.$$

But the value of the given fraction $\frac{ab}{b} = a$; hence, the fraction in each has been multiplied by b.

3. *Dividing the numerator or multiplying the denominator of a fraction by any quantity, divides the fraction by that quantity.*

For, let $\frac{ab^2}{b}$ be any fraction and b any integer. Then, dividing the numerator by b, we have

$$\frac{ab^2 \div b}{b} = \frac{ab}{b} = a,$$

and multiplying the denominator by b, we have

$$\frac{ab^2}{b \times b} = \frac{ab^2}{b^2} = a.$$

But the value of the given fraction $\frac{ab^2}{b} = ab$; hence, the fraction in each case has been divided by b.

4. *Multiplying or dividing both terms of a fraction by the same quantity does not change its value.*

For, let $\frac{ab}{b}$ be any fraction and b any integer. Then, multiplying both terms by b, we have

$$\frac{ab \times b}{b \times b} = \frac{ab^2}{b^2} = a.$$

Dividing both terms by b, we have

$$\frac{ab \div b}{b \div b} = \frac{a}{1} = a.$$

But the value of the given fraction $\frac{ab}{b} = a$; hence, in each case the value of the fraction is not changed.

5. *Changing all the signs of the terms of both numerator and denominator of a fraction does not change its value.*

For, let $\frac{a-b}{c+d}$ be any fraction, then, if we multiply both numerator and denominator by -1, we have

$$\frac{-a+b}{-c-d},$$

or the given fraction with the signs of its terms changed. But the value expressed must be the same as before, since multiplying both terms of a fraction by the same quantity does not change its value (Prin. 4, Art. 159).

SECTION XVI.

REDUCTION OF FRACTIONS.

160. Reduction of Fractions is the process of changing their form of expression without changing their value.

CASE I.

A Fraction to its Lowest Terms.

161. A fraction is in its **Lowest Terms,** when expressed in terms which are prime to each other.

162.—Ex. 1. Reduce $\frac{7a^2bc^2}{14abcd}$ to its lowest terms.

SOLUTION. Dividing both terms of the fraction by cancelling their common factors 7, a, b and c, which does not alter the value of the fraction (Prin. 4, Art. 159), we have its equal, $\frac{ac}{2d}$; which is in its lowest terms, since the terms are prime to each other.

$$\frac{7a^2bc^2}{14abcd}=\frac{ac}{2d}$$

2. Reduce to its lowest terms $\frac{ab}{bx}$. *Ans.* $\frac{a}{x}$.

3. Reduce $\frac{5ab}{10ab^2c}$ to its lowest terms. *Ans.* $\frac{1}{2bc}$.

163. Rule for Reducing a Fraction to its Lowest Terms.—*Cancel in both numerator and denominator all common factors, or divide both by their greatest common divisor.*

PROBLEMS.

1. Reduce $\frac{15x^3y}{3xy^2}$ to its lowest terms. *Ans.* $\frac{5x^2}{y}$.

2. Reduce $\frac{a^2bc}{5a^2b^2}$ to its lowest terms.

3. Reduce $\dfrac{x^2}{ax+x^2}$ to its lowest terms. *Ans.* $\dfrac{x}{a+x}$.

4. Reduce $\dfrac{16a^4b^2c}{20a^3b^3d}$ to its lowest terms.

5. Reduce $\dfrac{x^2-a^2}{x^4-a^4}$ to its lowest terms. *Ans.* $\dfrac{1}{x^2+a^2}$.

6. Reduce $\dfrac{ax+x^2}{ab^2+b^2x}$ to its lowest terms.

7. Reduce $\dfrac{ac-c^2}{a^2-c^2}$ to its lowest terms. *Ans.* $\dfrac{c}{a+c}$.

8. Reduce $\dfrac{x^2-1}{x^2-2x+1}$ to its lowest terms.

9. Reduce $\dfrac{x^2-4x+3}{4x^3-9x^2-15x+18}$ to its lowest terms.
Ans. $\dfrac{x-1}{4x^2+3x-6}$.

10. Reduce $\dfrac{a^3-b^3}{a^4-a^2b^2}$ to its lowest terms. *Ans.* $\dfrac{a^2+ab+b^2}{a^2(a+b)}$.

CASE II.

A Mixed Quantity to a Fraction.

164.—Ex. 1. Reduce $x+\dfrac{y}{2}$ to a fraction.

SOLUTION. Since in one there are 2 halves, in x there must be x times 2 halves, or $\dfrac{2x}{2}$; $\dfrac{2x}{2}$ and $\dfrac{y}{2}$ are $\dfrac{2x+y}{2}$, the fraction required.

$$x+\frac{y}{2}=\frac{2x}{2}+\frac{y}{2}$$

2. Reduce $4x-\dfrac{3x}{5a}$ to a fraction.

3. Reduce $b+\dfrac{ac}{a+c}$ to a fraction. *Ans.* $\dfrac{ab+bc+ac}{a+c}$.

165. Rule for Reducing a Mixed Quantity to a Fraction.— *Multiply the integral part by the denominator of the fraction, unite with the product the numerator of the fraction by its proper sign, and write the result over the denominator.*

PROBLEMS.

1. Reduce $2a+\dfrac{3-2x}{5}$ to a fraction. *Ans.* $\dfrac{10a+3-2x}{5}$.

2. Reduce $3a+4x-\dfrac{5}{2a}$ to a fraction.

3. Reduce $3b-\dfrac{ab+b^2}{2a}$ to a fraction.

SOLUTION.

$$3b-\frac{ab+b^2}{2a}=\frac{6ab-(ab+b^2)}{2a}=\frac{6ab-ab-b^2}{2a}=\frac{5ab-b^2}{2a}.$$

4. Reduce $5x+\dfrac{7x^2-4}{3x}$ to a fraction.

5. Reduce $1+\dfrac{a-b}{a+b}$ to a fraction. *Ans.* $\dfrac{2a}{a+b}$.

6. Reduce $x^2+x+1+\dfrac{2}{x-1}$ to a fraction.

7. Reduce $1-x+x^2-\dfrac{x^3}{1+x}$ to a fraction. *Ans.* $\dfrac{1}{1+x}$.

8. Reduce $a+b-\dfrac{a^2-b^2}{a+2b}$ to a fraction.

9. Reduce $x-y-\dfrac{x-y}{x+y}$ to a fraction.

10. Reduce $x^3+y-\dfrac{6x^3-4y^2}{6-4y}$ to a fraction. *Ans.* $\dfrac{6y-4x^3y}{6-4y}$.

CASE III.

A Fraction to an Integer or a Mixed Quantity.

166.—Ex. 1. Reduce $\frac{2x+y}{2}$ to a mixed quantity.

SOLUTION. The value of a fraction is the quotient arising from the division of its numerator by its denominator (Prin. 1, Art. 159). Performing the division denoted, we have the integral part x and the fractional part $\frac{y}{2}$, or $x+\frac{y}{2}$.

$$\frac{2x+y}{2}=x+\frac{y}{2}$$

2. Reduce $\frac{a^2+ac}{a}$ to an integer.

SOLUTION. Performing the division denoted, we have $a+c$, the integer required.

$$\frac{a^2+ac}{a}=a+c$$

3. Reduce $\frac{10a+3-2x}{5}$ to a mixed quantity.

Ans. $2a+\frac{3-2x}{5}$.

167. Rule for Reducing a Fraction to an Integer or a Mixed Quantity.—*Divide the numerator by the denominator.*

PROBLEMS.

1. Reduce $\frac{ab-2a^2}{ab}$ to a mixed quantity.

2. Reduce $\frac{a^2+x^2}{a-x}$ to a mixed quantity. *Ans.* $a+x+\frac{2x^2}{a-x}$.

3. Reduce $\frac{3a^2+5ax}{a+2x}$ to a mixed quantity.

4. Reduce $\frac{bc+d}{c}$ to a mixed quantity. *Ans.* $b+\frac{d}{c}$.

5. Reduce $\frac{x^3+1}{x-1}$ to a mixed quantity.

6. Reduce $\dfrac{a^2-2ax+x^2}{a-x}$ to an integral quantity.

7. Reduce $\dfrac{a^3-b^3}{a-b}$ to an integral quantity. *Ans.* a^2+ab+b^2.

8. Reduce $\dfrac{a^3b}{c^2d^3}$ to the form of an integral quantity.

$$\frac{a^3b}{c^2d^3}=a^3bc^{-2}d^{-3}$$

SOLUTION. Since the fraction may be regarded as an expression of division, we transfer the factors of the denominator to the numerator, which may be done by changing the signs of their exponents (Prin. 3, Art. 89), and obtain $a^3bc^{-2}d^{-3}$, the form required.

9. Reduce $\dfrac{x^3y}{a^{-2}y^{-2}}$ to the form of an integral quantity.

10. Reduce $\dfrac{a^2-2ab+b^2}{a^2-b^2}$ to the form of an integral quantity.

Ans. $(a-b)(a+b)^{-1}$.

CASE IV.

Dissimilar Fractions to Similar Fractions.

168. Similar Fractions are such as have the same denominator.

Thus, $\dfrac{xy}{a}$ and $\dfrac{bx^2}{a}$ are similar fractions.

169. Dissimilar Fractions are such as have different denominators.

Thus, $\dfrac{ab}{c}$ and $\dfrac{x+y}{a}$ are dissimilar fractions.

170. Fractions are said to be reduced to a **Common Denominator** when they are changed to equivalent fractions with denominators alike.

171.—Ex. 1. Reduce $\frac{c}{d}$ and $\frac{m}{n}$ to similar fractions.

$$\frac{c}{d}=\frac{c\times n}{d\times n}=\frac{cn}{dn}$$

$$\frac{m}{n}=\frac{m\times d}{n\times d}=\frac{dm}{dn}$$

SOLUTION. Since multiplying both terms of a fraction by the same quantity does not change its value, we multiply both terms of $\frac{c}{d}$ by n, the denominator of $\frac{m}{n}$, and have $\frac{cn}{dn}$; and multiply both terms of $\frac{m}{n}$ by d, the denominator of $\frac{c}{d}$, and have $\frac{dm}{dn}$. Hence, $\frac{c}{d}$ and $\frac{m}{n}$ equal $\frac{cn}{dn}$ and $\frac{dm}{dn}$, which are similar fractions.

2. Reduce $\frac{a}{2b}$, $\frac{5c}{3a}$ and $\frac{7a}{4b^2}$ to their least common denominator.

$$\frac{a}{2b}=\frac{a\times 6ab}{2b\times 6ab}=\frac{6a^2b}{12ab^2}$$

$$\frac{5c}{3a}=\frac{5c\times 4b^2}{3a\times 4b^2}=\frac{20b^2c}{12ab^2}$$

$$\frac{7a}{4b^2}=\frac{7a\times 3a}{4b^2\times 3a}=\frac{21a^2}{12ab^2}$$

SOLUTION. The least common multiple of the denominators is $12ab^2$, and as this contains only such factors as are required to compose the denominator, it must be the least common denominator of the fractions equivalent to the given fractions. Multiplying both terms of $\frac{a}{2b}$ by $6ab$, of $\frac{5c}{3a}$ by $4b^2$, and of $\frac{7a}{4b^2}$ by $3a$, we have $\frac{6a^2b}{12ab^2}$, $\frac{20b^2c}{12ab^2}$ and $\frac{21a^2}{12ab^2}$, which are the fractions reduced so as to have the denominator required.

3. Reduce $\frac{3}{4x}$, $\frac{ab}{6x^2}$ and $\frac{5}{12x^3}$ to their least common denominator. *Ans.* $\frac{9x^2}{12x^3}$, $\frac{2abx}{12x^3}$ and $\frac{5}{12x^3}$.

172. Rule for Reducing Dissimilar Fractions to Similar Fractions.—*Multiply both terms of one or more of the fractions by any quantity that will make the denominators alike. Or,—*

Multiply both terms of one or more of the fractions by any quantity that will give each a denominator equal to the least common multiple of the given denominators.

PROBLEMS.

1. Reduce $\frac{7b}{3a}$, $\frac{11ab}{12c}$ and $\frac{5ac}{8b}$ to similar fractions.

Ans. $\frac{56b^2c}{24abc}$, $\frac{22a^2b^2}{24abc}$ and $\frac{15a^2c^2}{24abc}$.

2. Reduce $\frac{2a}{3b}$, $\frac{3b}{4c}$ and $\frac{4c}{5x}$ to similar fractions.

3. Reduce $\frac{2a}{5b}$ aud $\frac{a+b}{3c}$ to similar fractions.

Ans. $\frac{6ac}{15bc}$ and $\frac{5ab+5b^2}{15bc}$.

4. Reduce $\frac{3b}{4}$, $\frac{2x}{3}$ and $a+\frac{4x}{5}$ to similar fractions.

5. Reduce $\frac{x+a}{3}$ and $\frac{2x+1}{a}$ to a common denominator.

Ans. $\frac{ax+a^2}{3a}$ and $\frac{6x+3}{3a}$.

6. Reduce $\frac{a}{2bx}$, $\frac{c}{6abxy}$ and $\frac{b}{3acx}$ to their least common denominator.

7. Reduce $x^2+\frac{a}{y}$ and $\frac{c}{ay-1}$ to their least common denominator.

Ans. $\frac{ax^2y^2+a^2y-x^2y-a}{ay^2-y}$ and $\frac{cy}{ay^2-y}$.

8. Reduce $\frac{a^2}{a+b}$, $\frac{ab}{a-b}$ and $\frac{3a^2-2ab}{a^2-b^2}$ to their least common denominator.

Ans. $\frac{a^2(a-b)}{a^2-b^2}$, $\frac{ab(a+b)}{a^2-b^2}$ and $\frac{3a^2-2ab}{a^2-b^2}$.

9. Reduce $\frac{a+x}{a-x}$ and $\frac{a-x}{a+x}$ to a common denominator.

Ans. $\frac{a^2+2ax+x^2}{a^2-x^2}$ and $\frac{a^2-2ax+x^2}{a^2-x^2}$.

SECTION XVII.

ADDITION OF FRACTIONS.

173. **Addition of Fractions** is the process of uniting two or more fractional quantities, to find their sum.

174. **Similar Fractions** may be added by finding the sum of their numerators, and placing it over their common denominator. Hence, fractions may be added by means of a common denominator.

Thus, $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$.

175.—Ex. 1. What is the sum of $\frac{a+c}{b}$ and $\frac{a-c}{b}$

$$\frac{a+c}{b}+\frac{a-c}{b}=\frac{a+c+a-c}{b}=\frac{2a}{b}$$

SOLUTION. The sum of $(a+c)$ divided by b, and $(a-c)$ divided by b, is the same as $(a+c)+(a-c)$ divided by b, or $\frac{a+c+a-c}{b}$, which is equal to $\frac{2a}{b}$.

2. What is the sum of $\frac{3a}{2}$ and $\frac{a}{3}$?

$$\frac{3a}{2}=\frac{3a\times 3}{2\times 3}=\frac{9a}{6}$$

$$\frac{a}{3}=\frac{a\times 2}{3\times 2}=\frac{2a}{6}$$

$$\frac{9a}{6}+\frac{2a}{6}=\frac{11a}{6}$$

SOLUTION. The sum of $3a$ divided by 2, and a divided by 3, is the same as the sum of $\frac{9a}{6}$ and $\frac{2a}{6}$, or $\frac{11a}{6}$.

3. What is the sum of $\frac{3x}{7}$ and $\frac{x-2}{5}$? *Ans.* $\frac{22x-14}{35}$.

176. **Rule for Adding Fractional Quantities.**—*Reduce the fractions, if necessary, to similar fractions; add their numerators, and write under the sum the common denominator.*

PROBLEMS.

1. Find the sum of $\frac{x}{3}$, $\frac{2x}{3}$, $\frac{5x}{3}$ and $\frac{2}{3}$. *Ans.* $\frac{8x+2}{3}$.

2. Find the sum of $\frac{3a^2}{2b}$, $\frac{2a}{5}$ and $\frac{3b}{7a}$. *Ans.* $\frac{105a^3+28a^2b+30b^2}{70ab}$.

3. Find the sum of $\frac{x}{a}$, $\frac{y}{b}$ and $\frac{z}{d}$.

4. Find the sum of $\frac{x}{5a}$, $\frac{3x}{5a}$, $\frac{4x}{5a}$ and $\frac{x-1}{5a}$. *Ans.* $\frac{9x-1}{5a}$.

5. Find the sum of $\frac{5x-9}{x+3}$ and $\frac{2x+11}{x-2}$.

6. Find the sum of $\frac{a}{a-2}$ and $\frac{a}{a-3}$. *Ans.* $\frac{2a^2-5a}{a^2-5a+6}$.

7. Add $\frac{x+y}{2}$ and $\frac{x-y-2z}{2}$.

8. Add $\frac{a^2}{a^2-1}$, $\frac{a}{a-1}$ and $\frac{a}{a+1}$.

9. Add $\frac{1+x^2}{1-x^2}$ and $\frac{1-x^2}{1+x^2}$. *Ans.* $\frac{2(1+x^4)}{1-x^4}$

177. When there are *Mixed Quantities* or *Integers, the fractions and integers may be added separately and the results united.*

1. Find the sum of $a+\frac{2bx}{c}$ and $b-\frac{3x^2}{a}$.

SOLUTION.

$$a+\frac{2bx}{c}+b-\frac{3x^2}{a}=a+b+\frac{2bx}{c}-\frac{3x^2}{a}$$
$$=a+b+\frac{2abx-3cx^2}{ac}$$

2. Find the sum of $2x$, $3x+\frac{2a}{5}$ and $x-\frac{8a}{9}$. *Ans.* $6x-\frac{22a}{45}$.

3. Find the sum of $11a$, $\frac{-b^2}{3}$ and $3a-\frac{b^2}{5}$.

4. Find the sum of $a+\frac{1}{a^2-b^2}$ and $2a-b+\frac{1}{(a-b)^2}$.

Ans. $3a-b+\frac{2a}{a^3-a^2b-ab^2+b^3}$.

5. Find the sum of $7b$, $\frac{2a}{3x^2}$ and $\frac{a+2x}{4x}$.

Ans. $7b+\frac{8a+3ax+6x^2}{12x^2}$.

SECTION XVIII.

SUBTRACTION OF FRACTIONS.

178. Subtraction of Fractions is the process of taking one fractional quantity from another, or of finding the difference between two fractional quantities.

179. Similar Fractions may be subtracted, one from another, by finding the difference of their numerators, and placing it over their common denominator.

Thus, $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$. Hence,

fractions may be subtracted by means of a common denominator.

180.—Ex. 1. Subtract $\frac{b}{c}$ from $\frac{a}{c}$.

SOLUTION. a divided by c, minus b divided by c, is the same as $(a-b)$ divided by c, or $\frac{a-b}{c}$

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

2. Subtract $\frac{b}{d}$ from $\frac{a}{c}$.

SOLUTION. Reducing the given fractions to similar fractions, we have $\frac{a}{c}=\frac{ad}{cd}$ and $\frac{b}{d}=\frac{bc}{cd}$, and $\frac{ad}{cd}-\frac{bc}{cd}$ is equal to $\frac{ad-bc}{cd}$.

$$\frac{a}{c}-\frac{b}{d}=\frac{ad}{cd}-\frac{bc}{cd}=\frac{ad-bc}{cd}$$

3. Subtract $\frac{5x}{4}$ from $\frac{14x}{8}$. *Ans.* $\frac{4x}{8}$.

4. Subtract $\frac{4c}{7y}$ from $\frac{5y}{6c}$. *Ans.* $\frac{35y^2-24c^2}{42cy}$.

181. Rule for Subtracting one Fraction from another.—*Reduce the fractions, if necessary, to similar fractions, and write the difference of the numerators over the common denominator.*

PROBLEMS.

1. From $\frac{2x}{5a}$ take $\frac{3c}{4a}$. *Ans.* $\frac{8x-15c}{20a}$.

2. From $\frac{a+b}{2}$ take $\frac{a-b}{2}$.

3. From $\frac{c}{a-b}$ take $\frac{c}{a+b}$. *Ans.* $\frac{2bc}{a^2-b^2}$.

4. From $\frac{9x+7}{4}$ take $\frac{6x+4}{5}$.

5. From $\frac{x+1}{x-1}$ take $\frac{x-1}{x+1}$. *Ans.* $\frac{4x}{x^2-1}$.

6. From $\frac{x}{m}$ take $\frac{y}{n}$.

7. From $\frac{1}{x+y}$ take $\frac{1}{x-y}$. *Ans.* $-\frac{2y}{x^2-y^2}$.

8. From $\frac{2x+y}{y}$ take $\frac{y-2x}{x}$. *Ans.* $\frac{2x^2+3xy-y^2}{xy}$.

9. From $\frac{x+y}{y}$ take $\frac{2x}{x+y}$.

10. From $\frac{7}{5a-4b}$ take $\frac{7}{5a+4b}$. *Ans.* $\frac{56b}{25a^2-16b^2}$.

182. When there are *Mixed Numbers, they may be expressed as fractions; or, if convenient, the fractional part of the subtrahend may be first subtracted, then the integral part, and the results united.*

1. From $7a+\frac{2a}{b}$ take $3a-\frac{a-3c}{b}$.

SOLUTION.

$$\left(7a+\frac{2a}{b}\right)-\left(3a-\frac{a-3c}{b}\right)=7a-3a+\frac{2a}{b}+\frac{a-3c}{b}$$
$$=4a+\frac{3a-3c}{b}$$

2. From $6a+\frac{3x-2a}{a}$ take $2a+\frac{4a-3x}{x}$.

3. From $5a-\frac{y-7}{4}$ take $2a+\frac{y-3}{12}$. *Ans.* $3a+2-\frac{y}{3}$.

4. What is the value of $8y^2-\frac{7y^2}{6}$?

5. What is the value of $\left(7x+\frac{x}{b}\right)-\left(3x-\frac{a-x}{b-x}\right)$?

Ans. $4x+\frac{ab-x^2}{b(b-x)}$.

SECTION XIX.

MULTIPLICATION OF FRACTIONS.

183. Multiplication of Fractions is the process of finding a product, when one or both of the factors are fractions.

CASE I.

A Fraction by an Integer.

184.—Ex. 1. Multiply $\frac{a}{b}$ by c.

SOLUTION. c times a divided by b is ac divided by b, or $\frac{ac}{b}$.

$$\frac{a}{b} \times c = \frac{ac}{b}$$

2. Multiply $\frac{a}{b^2}$ by b.

SOLUTION. b times a divided by b^2 is $\frac{ab}{b^2}$; or, since a fraction is multiplied by either multiplying its numerator or dividing its denominator, b times $\frac{a}{b^2}$ is $\frac{a}{b}$.

$$\frac{a}{b^2} \times b = \frac{ab}{b^2} = \frac{a}{b}$$

3. Multiply $\frac{a}{c}$ by d. *Ans.* $\frac{ad}{c}$.

4. Multiply $\frac{ab}{y}$ by y.

185. Rule for Multiplying a Fraction by an Integer.—*Multiply the numerator, or divide the denominator, by the integer.*

When there are factors common to the numerator and denominator, they can be cancelled.

PROBLEMS.

1. Multiply $\frac{a^2d}{bc}$ by c.

2. Multiply $\frac{ab^2}{dn}$ by m^2. *Ans.* $\frac{ab^2m^2}{dn}$.

3. Multiply $\frac{4ac}{b^3(a-c)}$ by ab^2c^{-1}

4. Multiply $\frac{5}{a-x}$ by a^2-x^2. *Ans.* $5(a+x)$.

5. Multiply $\frac{2a^2m}{a-n}$ by a^2-n^2.

186. When the multiplicand is a *Mixed Number, it may be expressed as a fraction, or the integral and the fractional parts may be multiplied separately, and the results united.*

1. Multiply $x+\frac{2b}{a}$ by $2a$.

SOLUTION.

$$\left(x+\frac{2b}{a}\right)\times 2a=2ax+\frac{4ab}{a}=2ax+4b.$$

2. Multiply $6a-c+\frac{d^2n}{bm}$ by $5m$. *Ans.* $30am-5cm+\frac{5d^2n}{b}$.

3. Multiply $a+1+\frac{1}{x}$ by $a-y$. *Ans.* $a^2+a-ay-y+\frac{a-y}{x}$.

4. Multiply $a+\frac{bc}{a}$ by $\frac{m}{n}$. *Ans.* $\frac{a^2m+bcm}{an}$

5. Multiply $3a+\frac{4b^2}{5c}$ by $3a-\frac{4b^2}{5c}$. *Ans.* $9a^2-\frac{16b^4}{25c^2}$.

CASE II.

An Integer or a Fraction, by a Fraction.

187.—Ex. 1. Multiply c by $\frac{a}{b}$.

SOLUTION. c multiplied by a is ac, and c multiplied by a divided by b must be ac divided by b, or $\frac{ac}{b}$.

$$c \times \frac{a}{b} = \frac{ac}{b}$$

2. Multiply $\frac{c}{d}$ by $\frac{a}{b}$.

SOLUTION. $\frac{c}{d}$ multiplied by a is $\frac{ac}{d}$, and $\frac{c}{d}$ multiplied by a divided by b, must be $\frac{ac}{d}$ divided by b, or $\frac{ac}{bd}$.

$$\frac{c}{d} \times \frac{a}{b} = \frac{ac}{bd}$$

3. Multiply $\frac{a}{b}$ by $\frac{c}{d}$. *Ans.* $\frac{ac}{bd}$.

4. Multiply $\frac{x}{5}$ by $\frac{4x}{9}$. *Ans.* $\frac{4x^2}{45}$.

188. Rule for Multiplying an Integer by a Fraction.—*Multiply the integer by the numerator of the fraction, and write the result over the denominator.*

189. Rule for Multiplying a Fraction by a Fraction.—*Multiply the numerators together for the numerator, and the denominators for the denominator, of the product.*

If there are mixed numbers, reduce them to fractions before multiplying.

Factors common to numerator and denominator can be cancelled.

PROBLEMS.

1. Multiply $\frac{1}{3a}$ by $\frac{2}{7a}$. *Ans.* $\frac{2}{21a^2}$.

2. Multiply $\frac{ab}{cd}$ by $\frac{3d}{4b}$.

3. Multiply $\frac{8}{7x}$ by $-\frac{3x}{5}$. *Ans.* $-\frac{24}{35}$.

4. Multiply $-\frac{3a}{5}$ by $-\frac{3a}{5}$.

5. Multiply $\frac{17a}{20b}$ by $\frac{5c}{34a^2}$. *Ans.* $\frac{c}{8ab}$.

6. Multiply $\frac{am+m^2}{ax^2}$ by $-\frac{x}{cm}$.

7. Multiply $\frac{a+x}{c+d}$ by $\frac{c-d}{a+x}$. *Ans.* $\frac{c-d}{c+d}$.

8. Multiply $\frac{a^2-9}{ab}$ by $\frac{b^2}{a+3}$.

9. Multiply $\frac{x^n}{y^m}$ by $\frac{x^m}{y^n}$. *Ans.* $\frac{x^{m+n}}{y^{m+n}}$.

10. Multiply $2a^2-3x$ by $\frac{3b}{y^{-1}}$.

11. What is the value of $5y^2 \times \left(y^2 - \frac{a}{3}\right)$? *Ans.* $5y^4 - \frac{5ay^2}{3}$.

12. What is the value of $\frac{ax}{x+a} \times \left(\frac{x}{a} - \frac{a}{x}\right)$?

13. What is the value of $3x \times \frac{x+1}{2a} \times \frac{x-1}{a+b}$?

Ans. $\frac{3x^3-3x}{2a^2+2ab} = \frac{3x(x^2-1)}{2a(a+b)}$.

14. What is the value of $\left(a + \frac{ab}{a-b}\right)\left(b - \frac{ab}{a+b}\right)$?

SECTION XX.

DIVISION OF FRACTIONS.

190. Division of Fractions is the process of finding a quotient, when the divisor or dividend, or both, are fractions.

CASE I.

A Fraction by an Integer.

191.—Ex. 1. Divide $\frac{ac}{b}$ by c.

$$\frac{ac}{b}\div c=\frac{a}{b}$$

SOLUTION. Dividing the numerator of a fraction divides the fraction (Prin. 3, Art. 159); hence, to divide $\frac{ac}{b}$ by c, we divide the numerator, ac, by c, and, writing the result over the denominator, b, we have $\frac{a}{b}$.

2. Divide $\frac{a}{b}$ by c.

$$\frac{a}{b}+c=\frac{a}{bc}$$

SOLUTION. Multiplying the denominator of a fraction divides the fraction (Prin. 3, Art. 159); hence, to divide $\frac{a}{b}$ by c, we multiply the denominator, b, by c, and, writing the result under the numerator, we have $\frac{a}{bc}$.

3. Divide $\frac{4a^2}{5xy}$ by $2a^2$. *Ans.* $\frac{2}{5xy}$.

4. Divide $\frac{3xy}{2bc^2}$ by $6y$. *Ans.* $\frac{x}{4bc^2}$.

192. Rule for Dividing a Fraction by an Integer. *Divide the numerator, or multiply the denominator, by the integer.*

When there are factors common to the numerator and the denominator, they can be cancelled.

PROBLEMS.

1. Divide $\frac{21x^2y}{7}$ by $3x^2$.

2. Divide $\frac{x^2-y^2}{ab^2}$ by $x-y$. *Ans.* $\frac{x+y}{ab^2}$.

3. Divide $\frac{a^2d+abd}{c}$ by ad.

4. Divide $\frac{a+ab}{1+b}$ by a. *Ans.* 1.

5. Divide $\frac{a^2-c^2}{1+x}$ by $a-c$.

6. Divide $\frac{60a^3}{7b}$ by $-9ab$. *Ans.* $-\frac{20a^2}{21b^2}$.

7. Divide $\frac{abx+b^2x}{y}$ by $a+b$.

8. Divide $\frac{1}{ac}$ by ac. *Ans.* $\frac{1}{a^2c^2}$.

9. Divide $\frac{a^2-c^2}{cd}$ by $a+2$.

10. Divide $\frac{6(a+b)}{7(b+c)}$ by $3(a+b)(b+c)$. *Ans.* $\frac{2}{7(b+c)^2}$

11. Divide $\frac{a+b+c}{11}$ by $3\,(x+y)$. *Ans.* $\frac{a+b+c}{33\,(x+y)}$.

12. Divide $\frac{ac+bd}{2(a+b)}$ by $2(a-b)$. *Ans.* $\frac{ac+bd}{4(a^2-b^2)}$.

CASE II.

An Integer or a Fraction, by a Fraction.

193. A fraction is **inverted** when the denominator is taken for the numerator, and the numerator for the denominator.

Thus, $\frac{a}{b}$ inverted is $\frac{b}{a}$; $\frac{x}{y}$ inverted is $\frac{y}{x}$.

194. A **Complex Fraction** is a fraction which has a fraction in one or both of its terms.

Thus, $\dfrac{\frac{a}{c}}{\frac{n}{m}}$ is a complex fraction, and is, also, an expression of the division of one fraction by another.

195.—Ex. 1. Divide a by $\frac{c}{d}$.

Solution. a divided by c is $\frac{a}{c}$, and a divided by c divided by d must be d times as great, or $\frac{ad}{c}$.

$$a \div \frac{c}{d} = \frac{a \times d}{c} = \frac{ad}{c}$$

2. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

Solution. Reduced to similar fractions, we have $\frac{a}{b} = \frac{ad}{bd}$, and $\frac{c}{d} = \frac{bc}{bd}$. The quotient, then, of the fractions is the same as that of their numerators, which is $\frac{ad}{bc}$.

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bc}$$

or,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c} = \frac{ad}{bc}$$

The same result is obtained by inverting the given divisor, and multiplying numerator by numerator and denominator by denominator.

3. Divide a^2b by $\frac{m}{n}$. *Ans.* $\frac{a^2bn}{m}$.

4. Divide $\frac{x}{6}$ by $\frac{2d}{y}$. *Ans.* $\frac{xy}{12d}$.

196. Rule for Dividing an Integer or a Fraction by a Fraction. —*Multiply the dividend by the divisor inverted.*

If there be mixed numbers, reduce them to fractions before dividing. Shorten the process of division as much as possible by cancelling.

PROBLEMS.

1. Divide $16a^2b^3$ by $\frac{8abc}{3}$. *Ans.* $\frac{6ab^2}{c}$.

2. Divide xy by $\frac{13bx}{ac}$. *Ans.* $\frac{acy}{13b}$.

3. Divide $\frac{5a}{b}$ by $\frac{7}{3b}$.

4. Divide $\frac{3ab}{c}$ by $\frac{4c}{a}$. *Ans.* $\frac{3a^2b}{4c^2}$.

5. Divide $\frac{5b}{ac}$ by $\frac{2a}{3c}$.

6. Divide $\frac{7x}{3a}$ by $\frac{3a}{5}$. *Ans.* $\frac{35x}{9a^2}$.

7. Divide $\frac{1}{x^2-y^2}$ by $\frac{1}{x-y}$.

8. Divide $\frac{8x^3}{x^3-y^3}$ by $\frac{4x^2}{x^2+xy+y^2}$. *Ans.* $\frac{2x}{x-y}$.

9. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+b^2}{x-b}$.

10. Divide $\frac{a+1}{6}$ by $\frac{2a}{3}$. *Ans.* $\frac{a+1}{4a}$.

11. Divide $5x^2-\frac{1}{5}$ by $x+\frac{1}{5}$.

12. Divide $a^3-\frac{1}{a^3}$ by $a-\frac{1}{a}$. *Ans.* $\frac{a^4+a^2+1}{a^2}=a^2+1+\frac{1}{a^2}$.

13. Divide $9a^2-\frac{16b^4}{25c^2}$ by $3a-\frac{4b^2}{5c}$.

Ans. $\frac{15ac+4b^2}{5c}=3a+\frac{4b^2}{5c}$.

197. The value of a *Complex Fraction* is found by performing the division indicated.

1. What is the value of $\dfrac{\frac{a}{c}}{\frac{b}{c}}$?

SOLUTION.

$$\frac{\frac{a}{c}}{\frac{b}{c}}=\frac{a}{c}\div\frac{c}{b}=\frac{a}{b}$$

2. What is the value of $\dfrac{\frac{5x^2}{7}}{4x}$?

3. What is the value of $\dfrac{\frac{2a^2-2y^2}{a+y}}{5cx}$? *Ans.* $\begin{cases}10acx-10cxy\\=10cx(a-y).\end{cases}$

4. Reduce $\dfrac{\frac{2m-2n}{3bm}}{\frac{5m-5n}{3mn}}$ to its simplest form.

5. Reduce $\dfrac{x+\frac{2x}{x-3}}{x-\frac{2x}{x-3}}$ to its simplest form. *Ans.* $\frac{x-1}{x-5}$.

6. Reduce $\dfrac{3a-\frac{2ab}{3c}}{\frac{a}{c}}$ to its simplest form. *Ans.* $\frac{9c-2b}{3}$.

SECTION XXI.

REVIEW PROBLEMS.

198.—Ex. 1. Reduce $\frac{cx+x^2}{ca^2+a^2x}$ to its lowest terms. *Ans.* $\frac{x}{a^2}$.

2. Reduce $a-\frac{b}{c}$ to a fraction. *Ans.* $\frac{ac-b}{c}$.

3. Reduce $\frac{ay+2y^2}{a+y}$ to a mixed number. *Ans.* $y+\frac{y^2}{a+y}$.

4. Reduce $\frac{m^3-n^3}{m-n}$ to an integer.

5. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d to a common denominator.

6. Add $x+\frac{x-2}{3}$ and $3x+\frac{2x-3}{4}$. *Ans.* $4x+\frac{10x-17}{12}$.

7. Subtract $\frac{2x+7}{8}$ from $\frac{3x+a}{5b}$. *Ans.* $\frac{24x+8a-10bx-35b}{0b}$

8. Multiply $\frac{x-1}{a+b}$ by x times $\frac{x+1}{a}$

9. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$. *Ans.* $x+\frac{b^2}{x}$.

10. What is the product of $b+\frac{1}{b}$, $b^2+\frac{1}{b^2}$ and $b-\frac{1}{b}$?

11. Multiply $3a+\frac{4b^2}{5d}$ by $3a-\frac{4b^2}{5d}$. *Ans.* $9a^2-\frac{16b^4}{26d^2}$.

12. Divide 12 by $\frac{(a+x)^2}{x}-a$. *Ans.* $\frac{12x}{a^2+ax+x^2}$.

Test Questions.

199.—1. What is a *Fraction?* What is the denominator of a fraction? The numerator? What are the terms of a fraction? What is an integer? A mixed number? The sign of a fraction? Mention some principles of fractions.

2. What is *Reduction* of fractions? When is a fraction in its lowest terms? What is the rule for reducing a fraction to its lowest terms? For reducing a mixed quantity to a fraction? A fraction to an integer or mixed number?

3. When are fractions said to have a *Common Denominator?* What are similar fractions? Dissimilar fractions? What is the rule for reducing dissimilar fractions to similar fractions?

4. What is *Addition* of fractions? How may similar fractions be added? What is the rule for adding fractional quantities?

5. What is *Subtraction* of fractions? How may similar fractions be subtracted? What is the rule for subtracting one fraction from another?

6. What is *Multiplication* of fractions? What is the rule for multiplying a fraction by an integer? For multiplying an integer or fraction by a fraction?

7. What is *Division* of fractions? What is the rule for dividing a fraction by an integer? For dividing an integer or fraction by a fraction?

8. What is a *Complex Fraction?* How is the value of a complex fraction found?

SECTION XXII.

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

200. The **Degree** of an equation containing but one unknown quantity is denoted by the highest power of that quantity.

Thus, $x+3=7$ is an equation of the first degree, and $x^2+x=12$ is an equation of the second degree.

201. A **Simple Equation** is an equation of the first degree, or one in which the highest power of the unknown quantity is the first power.

Thus, $x+4=18-6$ is a simple equation.

Clearing an Equation of Fractions.

202.—Ex. 1. If John has half as many apples as Edwin, and both have 12, what equation will express the number both have?

Solution. Let x equal the number of Edwin's apples; then half of x, or $\frac{x}{2}$, will equal John's number, and $x+\frac{x}{2}$ will equal the whole number of apples. But the whole number of apples is 12; hence, $x+\frac{x}{2}=12$ is the required equation.

2. Alice is half as old as Abbie, and the sum of their ages is 21. What equation will express the sum of their ages?

3. If each member of the equation $x+\frac{x}{2}=12$ be multiplied by 2, what will the equation become?

4. If each member of the equation $x+\frac{x}{3}=5$ be multiplied by 3, what will the equation become?

WRITTEN EXERCISES.

203.—Ex. 1. Clear $\frac{2x}{3}+12=\frac{4x}{5}+6$ of fractions.

Solution. Multiplying both members by 3, which will not affect their equality (Ax. 3, Art. 24), we have $2x+36=\frac{12x}{5}+18$, and multiplying both members of this equation by 5, we have $10x+180=12x+90$, which is clear of fractions.

$$\frac{2x}{3}+12=\frac{4x}{5}+6$$

$$2x+36=\frac{12x}{5}+18$$

$$10x+180=12x+90$$

Thus, each of the denominators was used in multiplying; but the same result would have been obtained by multiplying at once by their least common multiple, 15.

2. Clear the equation, $\frac{x}{a}=\frac{c}{b}+d$.

SOLUTION. Multiplying both members by ab, the least common multiple of the denominators of the fractions, we have $bx=ac+abd$.

$$\frac{x}{a}=\frac{c}{b}+d$$
$$bx=ac+abd$$

3. Clear the equation, $\frac{x}{a}=y-5$. *Ans.* $x=ay-5a$.

204. Rule for clearing an Equation of Fractions.—*Multiply both members of the equation by the least common multiple of the denominators, and reduce fractions to integers.*

When the sign before a fraction is $-$, on changing the fraction to an integer the sign of each term in the numerator must be changed, for the entire fraction is to be subtracted (Art. 158).

PROBLEMS.

1. Clear of fractions, $36-\frac{4x}{9}=8$. *Ans.* $324-4x=72$.

2. Clear of fractions, $x+\frac{x}{2}=175-2x$.
Ans. $2x+x=350-4x$.

3. Clear of fractions, $3x+\frac{5x}{4}=34$.

4. Clear of fractions, $\frac{x}{a}=b+c-d$. *Ans.* $x=ab+ac-ad$.

5. Clear of fractions, $\frac{x}{3}+\frac{x}{4}+\frac{x}{6}=9$.

6. Clear of fractions, $\frac{x-2}{2}+\frac{x-1}{3}=\frac{7}{6}$.
Ans. $3x-6+2x-2=7$.

7. Clear of fractions, $\frac{x}{6}-4=24-\frac{x}{8}$.

8. Clear of fractions, $x-\frac{x-4}{2}=11-\frac{x+2}{3}$.

9. Clear of fractions, $12\frac{(x+7)}{11}+8=5x$.

Ans. $12x+84+88=55x$.

10. Clear of fractions, $\frac{x-5a}{4a}-\frac{x-3a}{9}=\frac{a}{18}$.

11. Clear of fractions, $\frac{x-m^2}{4}+\frac{x-n^2}{6}=\frac{7mn}{12}$.

Ans. $3x-3m^2+2x-2n^2=7mn$.

SECTION XXIII.

SOLUTION OF SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

205. **Simple Equations** with one unknown quantity may be solved by so transposing the terms of the equation that the unknown quantity shall stand alone as one member; and the other member will then denote the value of the unknown quantity.

206. The **Root** of an equation is the value of the unknown quantity.

The root is **verified,** or the equation **satisfied,** when, this value being substituted in the given equation, the two members equal each other.

207.—Ex. 1. What is the value of x in the equation $\frac{x+15}{3}+1=x-14$?

SOLUTION. Clearing of the fractions, we have $x+15+3=3x-42$; transposing, we have $x-3x=-42-15-3$; uniting the terms, we have $-2x=-60$; and dividing by -2, the coefficient of x, we have $x=30$.

$$\frac{x+15}{3}+1=x-14$$
$$x+15+3=3x-42$$
$$x-3x=-42-15-3$$
$$-2x=-60$$
$$x=30$$

2. What is the value of x in the equation $\frac{2x-8}{4}=38+\frac{x}{6}$?

Ans. $x=120$.

208. Rule for solving Simple Equations with one unknown quantity.—*Clear the equation of fractions, if it have any.*

Transfer the terms, so that all those containing the unknown quantity shall be in the first member, and all the others shall be in the second member; unite the terms in each member, and divide both members by the coefficient of the unknown quantity.

PROBLEMS.

1. Given $x+\frac{x}{2}+\frac{x}{3}=11$, to find x. *Ans.* $x=6$.

2. Given $36-\frac{4x}{9}=8$, to find x.

3. Given $\frac{x}{5}+\frac{x}{3}=x-7$, to find x. *Ans.* $x=15$.

4. Given $\frac{x}{2}+\frac{x+1}{7}=x-2$, to find x.

5. Given $\frac{3x}{4}+\frac{180-5x}{6}=29$, to find x. *Ans.* $x=12$.

6. Given $\frac{x+1}{3}-\frac{3x-1}{5}=x-2$, to find x.

7. Given $\frac{x}{a}+\frac{x}{b}=c$, to find x.

Solution. Clearing of fractions, we have $bx+ax=abc$; factoring the first member, we have $(a+b)x=abc$; and dividing by $a+b$, the coefficient of x, we have $x=\frac{abc}{a+b}$

$$\frac{x}{a}+\frac{x}{b}=c$$
$$bx+ax=abc$$
$$(a+b)x=abc$$
$$x=\frac{abc}{a+b}$$

8. Given $\frac{a}{x}=\frac{b}{c}+\frac{d}{m}$, to find x.

9. Given $ax+b=cx+d$, to find x. *Ans.* $x=\frac{d-b}{a-c}$.

10. Given $3x+\frac{bx-d}{3}=x+a$, to find x.

11. Given $\frac{9x}{2}-\frac{3x-10}{5}+29=\frac{x+15}{5}+65$, to find x.

SOLUTION. Transposing and uniting similar terms, we have $\frac{9x}{2}-\frac{4x+5}{5}=36$; clearing of fractions, we have $45x-8x-10=360$; transposing and uniting, we have $37x=370$; and dividing by the coefficient of x, we have $x=10$.

$$\frac{9x}{2}-\frac{3x-10}{5}+29=\frac{x+15}{5}+65$$
$$\frac{9x}{2}-\frac{4x+5}{5}=36$$
$$45x-8x-10=360$$
$$37x=370$$
$$x=10$$

12. Given $56-\frac{6x}{8}=48-\frac{5x}{8}$, to find x.

13. Given $\frac{x-16}{4}=\frac{3x+4}{5}-\frac{14x-6}{4}$, to find x. *Ans.* $x=2$.

14. Given $\frac{2x}{3}+90=x+\frac{x-9}{3}+53$.

15. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x+19}{2}$, to find x. *Ans.* $x=9$.

16. Given $3x^2-10x=8x+x^2$, to find x.

SOLUTION. Dividing by x, we have the simple equation, $3x-10=8+x$; transposing and uniting, we have $2x=18$; and dividing by the coefficient of x, we have $x=9$.

$$3x^2-10x=8x+x^2$$
$$3x-10=8+x$$
$$2x=18$$
$$x=9$$

17. Given $ax^2+bx=cx^2+dx$, to find x. *Ans.* $x=\frac{d-b}{a-c}$.

18. Given $3ax^3-6abx^2=ax^3+3ax^2$.

19. Given $x^2-11x+30=x^2-5x+6$, to find x. *Ans.* $x=4$.

20. Given $12x^2+25x+7=12x^2+24x+12$, to find x.

21. Given $\frac{ab}{x}=bm+n+\frac{1}{x}$, to find x. *Ans.* $x=\frac{ab-1}{bm+n}$.

22. Given $x+ax-bc=ab+cx$, to find x.

23. Given $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-cx}$, to find x.
Ans. $x=\frac{ab+bc-ac}{a+b-c}$.

24. Given $\frac{ax+bx+cx}{a+b}=x+d$, to find x.

SECTION XXIV.

PROBLEMS PRODUCING SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

209.—Ex. 1. What number is that, the sum of whose fourth part and fifth part is 9?

SOLUTION. Let x denote the number; then $\frac{x}{4}$ will denote the fourth part, and $\frac{x}{5}$ the fifth part. Hence, by the conditions of the problem, we have $\frac{x}{4}+\frac{x}{5}=9$. Clearing of fractions, we have $5x+4x=180$; uniting terms, we have $9x=180$; and dividing, we have $x=20$.

$x=$ *the number*

$\frac{x}{4}=$ *the fourth part,*

$\frac{x}{5}=$ *the fifth part;*

$$\frac{x}{4}+\frac{x}{5}=9$$
$$5x+4x=180$$
$$9x=180$$
$$x=20$$

2. A certain sum of money increased by $\frac{2}{3}$ of it, and diminished by \$75, is \$175. What is the sum?

SOLUTION. Let x equal the sum; then $\frac{2x}{3}$ will denote $\frac{2}{3}$ of it, and by the conditions of the problem $x+\frac{2x}{3}-\$75$ is \$175. Transposing $-\$75$, and uniting it with \$250, we have $x+\frac{2x}{3}=\$250$. Clearing of fractions, we have $3x+2x=\$750$; uniting terms, we have $5x=\$750$; hence, $x=\$150$.

$$x = \textit{the sum};$$
$$\frac{2x}{3}=\frac{2}{3} \textit{ of the sum.}$$
$$x+\frac{2x}{3}-\$75=\$175$$
$$x+\frac{2x}{3}=\$250$$
$$3x+2x=\$750$$
$$5x=\$750$$
$$x=\$150$$

3. A line is 31 inches long; it is required to cut it into two parts, such that one part shall be 3 inches more than $\frac{3}{4}$ of the other part. What is the length of each part?

Ans. *16 inches, one part; 15 inches, the other.*

4. A man bequeathed \$161 to two sons; to one he left \$17 more than half as much as to the other. What did he leave to each?

5. Out of a cask of molasses, $\frac{1}{3}$ of which had leaked away, 21 gallons were drawn, and then the cask was found to be half full. What was the capacity of the cask?

SOLUTION. Let x denote the capacity of the cask in gallons; then $\frac{x}{3}$ will denote what leaked away, and by the conditions of the problem $21+\frac{x}{3}$ will be one-half of the capacity of the cask, or $\frac{x}{2}$. Clearing of fractions, we have $126+2x=3x$; transposing and uniting terms, we have $-x=-126$; whence, $x=126$.

$$x = \textit{capacity of the cask};$$
$$\frac{x}{3} = \textit{what leaked away.}$$
$$21+\frac{x}{3}=\frac{x}{2}$$
$$126+2x=3x$$
$$-x=-126$$
$$x=126$$

6. When I had spent from my purse \$70 more than $\frac{2}{5}$ of the money there had been in it, I found only $\frac{1}{4}$ of the money left. How much money had I at first?

7. What is a man's age, if $\frac{3}{4}$ of his age less 10 years is just $\frac{5}{8}$ of his age? *Ans. 80 years.*

8. A cistern was $\frac{2}{9}$ full of water. After 17 hogsheads had run in, it was found to be $\frac{3}{8}$ full. What number of hogsheads is its capacity?

9. A alone can do a piece of work in 9 days, and B alone can do it in 12 days. In what time will they do it if they work together?

SOLUTION. Let x denote the time in which they can do the work by working together; as A does $\frac{1}{9}$ of the work in 1 day, in x days he will do $\frac{x}{9}$ of the work, and in like manner B will do $\frac{x}{12}$ of the work.

Denoting the entire work by 1, we have, by the conditions of the problem, $\frac{x}{9}+\frac{x}{12}=1$. Clearing of fractions, we have $4x+3x=36$; whence, $x=5\frac{1}{7}$.

x = *the time;*

$\frac{x}{9}$ = *what A can do;*

$\frac{x}{12}$ = *what B can do.*

$$\frac{x}{9}+\frac{x}{12}=1$$

$$4x+3x=36$$

$$7x=36$$

$$x=5\frac{1}{7}$$

10. A man can do a piece of work in 8 days, and his son can do the same in 3 times as many days. In what time can they do it by working together? *Ans. 6 days.*

11. A certain canister of tea will last a woman 12 days, and will last her husband $\frac{2}{3}$ of that time. How long would it last them, using it together?

12. A cistern can be filled with water by means of one pipe in a hours, and by means of another in b hours. In how many hours could the cistern be filled by both pipes running together? *Ans.* $\frac{ab}{a+b}$ *hours.*

13. A cistern can be filled by means of one pipe in 6 hours, and by means of another pipe in 8 hours; and it can be emptied by a tap in 12 hours, if the two pipes are closed. In

what time will the cistern be filled, if the pipes and the tap are all open?

14. A gentleman gave in charity \$46, a part of which he distributed in equal portions to 5 poor men, and the rest in equal portions to 7 poor women. It was found that a man and a woman had together \$8. How much was given to each man, and how much to each woman?

15. A has \$900 and B \$700. What sum must A give to B in order that B may have $\frac{9}{7}$ as much as A? *Ans.* *\$200.*

16. A man can row 6 miles an hour with the current of a river, and 4 miles an hour against the current, his rowing in each direction being uniform. How far can he go in order that the time between leaving and returning to the place from which he started shall be $2\frac{1}{2}$ hours?

SOLUTION.

x = *number of miles out;*

$\frac{x}{6}$ = *time with the current;*

$\frac{x}{4}$ = *time against the current.*

$$\frac{x}{6}+\frac{x}{4}=2\frac{1}{2}$$

$x=6$, *number of miles out.*

17. A person walked to the top of a mountain at the uniform rate of $2\frac{1}{3}$ miles an hour, and without stopping walked down again by the same way at the uniform rate of $3\frac{1}{2}$ miles an hour. He walked 5 hours. How far did he walk?

Ans. *7 miles up, and 7 miles down.*

18. If a person have only a hours at his disposal, how far cau he ride in a coach which travels b miles an hour, and return home in the time, walking back at the rate of c miles an hour?

Ans. $\frac{abc}{b+c}$ *miles.*

19. A starts from a certain place, and travels at the rate of 7 miles in 5 hours; B starts from the same place 8 hours after A, and travels in the same direction at the rate of 5 miles

in 3 hours. How far will A travel before he is overtaken by B?

SOLUTION. Let x denote the number of hours which A travels before he is overtaken; then B travels $x-8$ hours. A travels 7 miles in 5 hours; therefore, he travels $\frac{7}{5}$ of a mile in one hour, and $\frac{7}{5}x$ miles in x hours. B travels $\frac{5}{3}$ of a mile in one hour, and $\frac{5}{3}(x-8)$ miles in $x-8$ hours. But, when B overtakes A, they have travelled the same number of miles; hence, we have $\frac{5}{3}(x-8)=\frac{7x}{5}$; whence, $x=50$, and $\frac{7x}{5}=70$.

$$x = \textit{the number of hours A travels;}$$
$$x-8 = \textit{the number of hours B travels;}$$
$$\frac{7x}{5} = \textit{the distance A travels.}$$
$$\frac{5}{3}(x-8) = \textit{the distance B travels.}$$
$$\frac{5}{3}(x-8)=\frac{7x}{5}$$
$$25x-200=21x$$
$$4x=200$$
$$x=50$$
$$\frac{7x}{5}=70, \textit{ the distance A travels.}$$

20. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off, running at the rate of 8 miles an hour. How many miles can the ship run before it is overtaken?

Ans. 72.

21. A clock has two hands turning on the same center. The swifter makes a revolution every 12 hours, and the slower every 16 hours. In what time will the swifter gain one complete revolution on the slower?

22. A man loaned \$1000, a part at 4 per cent., and the rest at 5 per cent. The whole yearly interest received was \$44. What sum was lent at 4 per cent.?

SOLUTION.

$$x = \textit{the sum lent at 4 per cent.}$$
$$\$1000-x = \textit{the sum lent at 5 per cent.}$$
$$\frac{4x}{100} = \textit{yearly interest on sum lent at 4 per cent.}$$
$$\frac{5(\$1000-x)}{100} = \textit{yearly interest on sum lent at 5 per cent.}$$
$$\frac{4x}{100}+\frac{5(\$1000-x)}{100}=\$44, \textit{ or } 4x+\$5000-5x=\$4400$$
$$x=\$600, \textit{ the sum lent at 4 per cent.}$$

23. A person lent \$900, a part at the rate of 4 per cent., and a part at the rate of 5 per cent., and he received equal sums as interest from the two parts. How much did he lend at each rate?

24. A manufacturer adds to the cost price of goods 20 per cent., to give the selling price. Afterward, to effect a rapid sale, he deducts from the selling price of each article a discount of 10 per cent., and then obtains on each article a profit of \$8. What was the cost price of each article? *Ans. \$100.*

25. A and B join capital for speculation, B contributing \$250 more than A. If their profits amount to 10 per cent. on their joint capital, B's share of them is equal to 12 per cent. of A's capital. How much does each contribute?

26. A man has a number of coins which he tries to arrange in the form of a square. At the first attempt he has 130 over. When he increases the side of the square by 3 coins, he has only 31 over. How many coins has he?

SOLUTION.

x = *the number of coins in each row at the first attempt;*

x^2 = *the number of coins in the square at first attempt;*

x^2+130 = *the whole number of coins.*

$(x+3)^2$ = *the number of coins in the square at second attempt;*

$(x+3)^2+31$ = *the whole number of coins.*

$$x^2+130=(x+3)^2+31$$
$$x^2+130=x^2+6x+9+31$$
$$6x=90$$
$$x=15$$

$x^2+130=355$, *the number of coins he has.*

27. A colonel wishes to arrange his men in a solid square. In the first formation he has 39 men over. When he increases the side of the square by 1 man, he wants 50 men to complete the square. How many men has he?

28. A regiment was drawn up in a solid square: when some time afterward it was again drawn up in a solid square, it was

found that there were 5 men fewer on a side. In the interval 295 men had been removed from the field. What was the original number of men in the regiment? *Ans. 1024 men.*

29. At a public meeting a resolution was carried by a majority of 9; but if $\frac{1}{6}$ of those who voted for it had voted against it, it would have been lost by 3 votes. How many voted?

SECTION XXV.

SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES.

210. Independent Equations are such as express essentially different conditions, or cannot be reduced to the same form.

Thus, $\frac{x}{2}-\frac{y}{4}=6$ and $6x-4y=4$ are independent equations; but $x+y=7$ and $2x+2y=14$ are not independent equations, because one can be directly obtained from the other.

211. Simultaneous Equations are any two or more equations in which the same unknown quantities express the same values.

Thus, in each of the equations $x+y=10$ and $5x-2y=8$ the value of x is 4, and of y is 6.

When two or more unknown quantities in an equation are to be determined, there must be as many independent and simultaneous equations as there are unknown quantities, and from these must be deduced a single equation containing only one unknown quantity.

ELIMINATION.

212.—Ex. 1. If $x+y$ be added to $x-y$, what expression will represent the result?

2. If $2x+y$ be added to $x-y$, what will represent the sum?

3. If $x+2y$ be added to $x-2y$, what will represent the sum?

4. If the equation $x-y=2$ be added to the equation $x+y=16$, what expression will represent the sum?

5. If the value of x in the equation $x+y=18$ is 9, what is the value of y?

6. If the equation $3x-2y=7$ be added to the equation $x+2y=13$, what will represent the sum? What is the value of x? What is the value of y?

7. If $x-2y$ be taken from $2x-2y$, what will represent the result?

8. If the equation $x-2y=1$ be taken from $2x-2y=8$, what will represent the result? What is the value of x? Of y?

Definition.

213. Elimination is the process of deducing from given simultaneous equations one or more equations, containing together fewer unknown quantities than the given equations.

CASE I.

By Comparison.

214. Elimination by Comparison consists in finding an expression of the value of the same unknown quantity in each of the equations, and in forming a new equation from those values.

215.—Ex. 1. Given $x+3y=9$ and $3x+2y=13$, to find x and y.

SOLUTION. Transposing $3y$ in equation (1), we obtain (3). Transposing $2y$ in (2) and dividing by 3, we obtain (4). Placing these two values of x equal to each other, we obtain (5). Clearing of fractions, we obtain (6). Uniting and transposing, we obtain (7). Dividing by 7, we obtain (8), or $y=2$. Substituting 2 for y in 3, we obtain (9). Uniting, we obtain (10), or $x=3$.

$$x+3y=9 \quad (1)$$
$$3x+2y=13 \quad (2)$$
$$x=9-3y \quad (3)$$
$$x=\frac{13-2y}{3} \quad (4)$$
$$\frac{13-2y}{3}=9-3y \quad (5)$$
$$13-2y=27-9y \quad (6)$$
$$7y=14 \quad (7)$$
$$y=2 \quad (8)$$
$$x=9-6 \quad (9)$$
$$x=3 \quad (10)$$

2. Given $5x+2y=45$ and $4x+y=33$, to find the values of x and y. *Ans.* $x=7$; $y=5$.

216. Rule for Elimination by Comparison.—*Find an expression for the value of the same unknown quantity in each of the equations. Form a new equation by placing these values equal to each other, and solve the equation.*

PROBLEMS.

1. Given $x+y=10$ and $2x-3y=5$, to find x and y.

Ans. $x=7$; $y=3$.

2. Given $9x-4y=8$ and $5x+3y=41$, to find x and y.

3. Given $3x+7y=99$ and $11x-5y=87$, to find x and y.

Ans. $x=12$; $y=9$.

4. Given $9x-4y=8$ and $13x+7y=101$, to find x and y.

5. Given $\frac{x}{5}+\frac{y}{6}=18$ and $\frac{x}{2}-\frac{y}{4}=21$, to find x and y.

Ans. $x=60$; $y=36$.

6. Given $3x+\frac{y}{3}=36$ and $6y-2x=32$, to find x and y.

7. Given $2x-\frac{y-3}{5}=4$ and $3y+\frac{x-2}{3}=9$, to find x and y.

Ans. $x=2$; $y=3$.

8. Given $\frac{y-x}{y}=\frac{3}{8}$ and $\frac{x}{y-2}=\frac{5}{6}$, to find x and y.

9. Given $\frac{x+y}{3}+x=15$ and $\frac{x-y}{5}+y=6$, to find x and y.

Ans. $x=10$; $y=5$.

10. Given $\frac{x}{3}+3y=7$ and $\frac{4x-2}{5}=3y-4$, to find x and y.

CASE II.

By Substitution.

217. Elimination by Substitution consists in finding an expression of the value of one of the unknown quantities in one of the equations, and substituting this value in another equation.

218.—Ex. 1. Given $x+3y=9$ and $3x+2y=13$, to find x and y.

SOLUTION. From equation (1) by transposing $3y$, we obtain (3). Substituting this value of x in equation (2), we obtain (4). Uniting, we obtain (5). Dividing by -7, we have (6), or $y=2$. Substituting the value of y in equation (3), we obtain (7), or $x=3$.

$$x+3y=9 \quad (1)$$
$$3x+2y=13 \quad (2)$$
$$x=9-3y \quad (3)$$
$$27-9y+2y=13 \quad (4)$$
$$-7y=-14 \quad (5)$$
$$y=2 \quad (6)$$
$$x=9-6=3 \quad (7)$$

2. Given $8x+7y=100$ and $12x-5y=88$, to find x and y.
Ans. $x=9$; $y=4$.

219. Rule for Elimination by Substitution.—*Find an expression for the value of one of the unknown quantities in one of the equations. Substitute this value for the same unknown quantity in the other equation, and solve the equation.*

PROBLEMS.

1. Given $4x+9y=46$ and $8x-13y=30$, to find x and y.
Ans. $x=7$; $y=2$.

2. Given $7x+5y=33$ and $13x-11y=41$, to find x and y.

3. Given $x+2y=15$ and $5x-19y=17$, to find x and y.
Ans. $x=11$; $y=2$.

4. Given $3x+\frac{y}{3}=36$ and $6y-2x=32$, to find x and y.

5. Given $7x+2y=30$ and $5x+3y=34$, to find x and y.
Ans. $x=2$; $y=8$.

6. Given $\frac{x}{2}+\frac{y}{3}=8$ and $\frac{x}{3}-\frac{y}{2}=1$, to find x and y.

7. Given $\frac{6x}{4}-\frac{y}{3}=11$ and $\frac{3x}{5}+\frac{7y}{4}=2x+7$, to find x and y.

8. Given $\frac{9x}{x}-\frac{4}{y}=7$ and $\frac{18}{x}+\frac{20}{y}=16$, to find x and y.

Ans. $x=3$; $y=2$.

9. Given $4x-\frac{y}{2}=11$ and $2x-4y=0$, to find x and y.

10. Given $\frac{2x+3y}{4}=5$ and $2x=\frac{54-8y}{3}$, to find x and y.

CASE III.

By Addition or Subtraction.

220. Elimination by Addition or Subtraction consists in adding two equations, or subtracting one equation from another, when the coefficient of the same unknown quantity in each is the same, or has been made the same.

221.—Ex. 1. Given $8x+7y=100$ and $12x-5y=88$, to find x and y.

Solution. Multiplying equation (1) by 5, and equation (2) by 7, we obtain equations (3) and (4), in which the coefficients of y are the same. Adding equations (3) and (4), we obtain (5). Uniting, we obtain (6). Dividing by 124, we obtain (7), or $x=9$. Substituting 9 for x in (1), we obtain (8). Transposing and uniting, we obtain (9). Dividing by 7, we obtain (10), or $y=4$.

$$8x+7y=100 \quad (1)$$
$$12x-5y=88 \quad (2)$$
$$40x+35y=500 \quad (3)$$
$$84x-35y=616 \quad (4)$$
$$40x+84x=500+616 \quad (5)$$
$$124x=1116 \quad (6)$$
$$x=9 \quad (7)$$
$$72+7y=100 \quad (8)$$
$$7y=28 \quad (9)$$
$$y=4 \quad (10)$$

2. Given $x+3y=9$ and $3x+2y=13$, to find x and y.

SOLUTION. Multiplying equation (1) by 3, we obtain (3). Subtracting equation (2) from equation (3), we obtain (4). Dividing by 7, we obtain (5), or $y=2$. Substituting 2 for y in (1), we obtain (6). Transposing and uniting, we obtain (7), or $x=3$.

$$x+3y=9 \quad (1)$$
$$3x+2y=13 \quad (2)$$
$$3x+9y=27 \quad (3)$$
$$7y=14 \quad (4)$$
$$y=2 \quad (5)$$
$$x+6=9 \quad (6)$$
$$x=3 \quad (7)$$

3. Given $7x+3y=42$ and $8y-2x=50$, to find x and y.

Ans. $x=3$; $y=7$.

222. Rule for Elimination by Addition and Subtraction.—*Multiply or divide one or both of the equations, if necessary, so that one of the unknown quantities shall have the same coefficient in both equations. Then, if the signs of the terms containing this quantity are alike, subtract one equation from the other; if unlike, add the two equations.*

Of the different methods of elimination, each has its advantages in application to particular problems. When the coefficient of one of the unknown quantities is 1, the method by substitution is often the most convenient. In general, however, elimination by addition or subtraction, as it does not give rise to fractions, is the most simple, and is therefore to be preferred.

PROBLEMS.

1. Given $6x-7y=42$ and $7x-6y=75$, to find x and y.

Ans. $x=21$; $y=12$.

2. Given $3x-4y=18$ and $3x+2y=0$, to find x and y.

3. Given $\frac{x}{2}-\frac{y}{4}=20$ and $\frac{x+y}{5}-\frac{2y-x}{4}+\frac{x}{3}=35$, to find x and y.

Ans. $x=60$; $y=40$.

4. Given $5x+4y=22$ and $2x+3y=13$, to find x and y.

5. Given $3x+7y=79$ and $2y-\frac{1}{2}x=9$, to find x and y.
Ans. $x=10$; $y=7$.

6. Given $3x+\frac{y}{3}=36$ and $6x-2y=48$, to find x and y.

7. Given $\frac{2x+3y}{5}=10-\frac{y}{3}$ and $\frac{4y-3x}{6}=\frac{3x}{4}+1$, to find x and y.
Ans. $x=4$; $y=9$.

8. Given $\frac{1-3x}{7}+\frac{3y-1}{5}=2$ and $\frac{3x+y}{11}+y=9$, to find x and y.

9. Given $\frac{x}{a}+\frac{y}{b}=2$ and $bx-ay=0$, to find x and y.
Ans. $x=a$; $y=b$.

10. Given $2x+\frac{y-2}{5}=21$ and $4y+\frac{x-4}{6}=29$, to find x and y.

223. In finding the values of unknown quantities in the following equations, the learner may use any of the methods of elimination which may be deemed most convenient.

1. Given $10x=69+\frac{y}{5}$ and $10y=49+\frac{x}{7}$, to find x and y.
Ans. $x=7$; $y=5$.

2. Given $6x+5y=70$ and $\frac{3x}{2}+\frac{y}{4}=9\frac{1}{2}$, to find x and y.

3. Given $6x+8y=52$ and $3x+7y=32$, to find x and y.
Ans. $x=6$; $y=2$.

4. Given $\frac{1}{7x}+\frac{y}{9}=11$ and $\frac{1}{9x}+\frac{y}{2}=16$, to find x and y.

5. Given $\frac{x+1}{y}=\frac{1}{3}$ and $\frac{x}{y+1}=\frac{1}{4}$, to find x and y.
Ans. $x=4$; $y=15$.

6. Given $3x+8y=60$ and $3y+6x=42$, to find x and y.

7. Given $\frac{2x}{3}-\frac{2y}{5}=-2$ and $\frac{x}{2}+\frac{y}{3}=8$, to find x and y.
Ans. $x=6$; $y=15$.

8. Given $\frac{2x}{3}+\frac{3y}{2}=16\frac{1}{6}$ and $\frac{3x}{2}-\frac{2y}{3}=16\frac{1}{6}$, to find x and y.

9. Given $\frac{7x}{4}+\frac{5y}{8}=20$ and $\frac{3x}{5}+\frac{7y}{4}=2x-7$, to find x and y.

10. Given $5x+11y=146$ and $11x+5y=110$, to find x and y.
Ans. $x=5$; $y=11$.

11. Given $\frac{2x-y}{4}+2y=6$ and $4y-\frac{x+10}{3}=3$.

12. Given $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$ and $\frac{1}{x}-\frac{1}{y}=\frac{1}{b}$, to find x and y.
Ans. $x=\frac{2ab}{a+b}$; $y=-\frac{2ab}{a-b}$.

13. Given $ax=by$ and $x+y=c$, to find x and y.

14. Given $4x+y=11$ and $\frac{y}{5x}=\frac{7x-y}{3x}-\frac{23}{15}$, to find x and y.

15. Given $12x-7y=3$ and $9x+5y=33$, to find x and y.
Ans. $x=2$; $y=3$.

16. Given $\frac{7x-9y}{3}=8$ and $\frac{5x+5y}{4}=16$, to find x and y.

17. Given $7\frac{1}{2}y+13\frac{1}{3}z=660$ and $8\frac{1}{3}y+5\frac{1}{2}z=398$, to find y and z.
Ans. $y=24$; $z=36$.

SECTION XXVI.

PROBLEMS PRODUCING SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES.

224.—Ex. 1. My herd of cattle is such that if the number of cows be added to four times the number of the oxen, the sum is 29; and if the number of oxen be added to four times the number of cows, the sum is 26. What is the number of each?

x = *the number of cows;*
y = *the number of oxen.*

$$
\begin{array}{rl}
x+4y=29 & (1)\\
y+4x=26 & (2)\\
4y+16x=104 & (3)=(2)\times 4\\
15x=75 & (4)=(3)-(1)\\
x=5 & (5)=(4)\div 15\\
y+20=26 & (6)=(2)\ \textit{after substitution.}\\
y=6 & (7)=(6)\ \textit{after transposition and reduction.}
\end{array}
$$

Solution. Let x denote the number of cows, and y the number of oxen. Then, by the conditions of the problem, we have $x+4y=29$ and $y+4x=26$. Multiplying equation (2) by 4, we obtain (3). Subtracting equation (1) from (3), we obtain (4). Dividing by 15, we obtain (5), or $x=5$. Substituting the value of x in equation (2), we obtain (6). Transposing and uniting, we obtain (7), or $y=6$.

2. A's and B's ages are such that if twice A's age be added to three times B's, the sum will be 140; and if B's be added to four times A's, the sum will be 130. What are their ages?

3. A farmer sold to one person 9 horses and 7 cows for $300, and to another, at the same prices, 6 horses and 13 cows for the same sum. What were the prices?

Ans. A horse, $24; a cow, $12.

4. There are two numbers such that if twice the less be added to the greater, the sum will be 18; and if three times the greater be diminished by the less, the remainder will be 19. Find the numbers. *Ans. The greater, 8; the less, 5.*

5. Two pieces of cloth, measuring together 57 yards, were sold for \$36. The first was valued at \$.75 per yard, and the second at \$.50. What was the number of yards in each piece? *Ans. In one piece, 30; in the other, 27.*

6. A purse contains dimes and dollars. Add a dime, and then there will be twice as many dimes as dollars. Add a dollar to the original contents of the purse, and then there will be more dimes than dollars by 2. What number of dimes and dollars does the purse contain?

SOLUTION. Let x denote the number of dollars, and y the number of dimes. Then, by the conditions of the problem, we have $y+1=2x$ and $y=x+1+2$. Transposing equation (1), we obtain (3). Comparing equations (2) and (3), we obtain (4). Transposing and uniting, we obtain (5), or $x=4$. Substituting the value of x in equation (3), we obtain (6), or $y=7$.

$x=$ the number of dollars;
$y=$ the number of dimes.

$$y+1=2x \qquad (1)$$
$$y=x+1+2 \qquad (2)$$
$$y=2x-1 \qquad (3)$$
$$2x-1=x+1+2 \qquad (4)$$
$$x=4 \qquad (5)$$
$$y=8-1=7 \qquad (6)$$

7. The ages of two persons are such that if to the sum of their ages 22 be added, the sum will be double the age of the elder, and if 1 be taken from the difference of their ages, the remainder will be the age of the younger. How old is each?

8. A says to B, "If you will give me half of your money, I shall have \$100." B replies, "I shall have \$100 if you will give me a third of your money." How much had each?

9. A liberty pole consists of two parts; one-third of the lower part, added to one-sixth of the upper part, is equal to 28 feet; and five times the lower part, diminished by six times the upper part, is equal to 12 feet. What is the height of the pole? *Ans. 108 feet.*

10. The numbers in two opposing armies are such that the sum of both is 21110; and twice the number in the greater army, added to three times the number in the less, is 52219. What is the number in the greater army? *Ans. 11111.*

11. There are two numbers such that the first added to half the second gives 35; the second added to half the first gives 40. What are the numbers?

12. A certain number expressed by two figures is equal to six times the sum of the digits, and if 117 be subtracted from three times the number, the order of the figures is reversed. What is the number?

SOLUTION.

x = *the number expressed by the tens' figure.*
y = *the number expressed by the units' figure.*
$10x+y$ = *the number expressed by the figures.*
$10y+x$ = *the number expressed by the figures reversed.*

$$10x+y=6(x+y) \qquad (1)$$
$$3(10x+y)-117=10y+x \qquad (2)$$
$$4x=5y \qquad (3)=(1) \textit{ simplified.}$$
$$29x-7y=117 \qquad (4)=(2) \textit{ simplified.}$$
$$-y=-\frac{4x}{5}, \textit{ value of } -y \textit{ from } (3)$$
$$-y=\frac{117-29x}{7}, \textit{ value of } -y \textit{ from } (4)$$
$$-\frac{4x}{5}=\frac{117-29x}{7}, \textit{ two values of } -y \textit{ are equal; whence,}$$
$$117x=585;\ \textit{whence, } x=5.$$

Substituting value of x in (3), we have $y=4$. Hence, 54 is the number.

13. There is a number expressed by two figures, the sum of whose digits is 13, and if 27 be subtracted from the number, the digits will be inverted. What is that number?

14. A says to B, "Give me $1 of your money, and I shall have twice as much as you will have left." "Yes," says B, "but give me $1 of your money, and I shall have three times as much as you will have left." How much money has each?

Ans. A, $\$2\frac{1}{5}$; B, $\$2\frac{3}{5}$.

15. Two numbers are such that if one were increased by 18, it would be double the other, and if the second were diminished by 11, it would be one-third of the former. What are the numbers?

16. A fraction becomes equal to 2, when 7 is added to its numerator, and equal to 1, when 1 is subtracted from its denominator. What is the fraction?

SOLUTION.

$x=$ *the numerator.*

$y=$ *the denominator.*

$\frac{x}{y}=$ *the fraction.*

Then, by the conditions, $\begin{cases} \frac{x+7}{y}=2 & (1) \\ \frac{x}{y-1}=1 & (2) \end{cases}$

Reducing (1), $x=2y-7$ (3)

Reducing (2), $x=y-1$ (4)

From (3) *and* (4), $2y-7=y-1$ (5)

whence, $y=6$ (6)

and, $x=6-1=5$ (7)

From (6) *and* (7), $\frac{x}{y}=\frac{5}{6}$

17. What fraction is that whose value will be $\frac{1}{2}$, if its numerator be increased by 3; and whose value will be $\frac{1}{5}$, if its denominator be diminished by 3?

18. A person has two horses, and a saddle worth \$50. If the saddle be sold with the first horse, it will make his value double that of the second; but if it be sold with the second, it will make his value three times that of the first. What is the value of each horse? *Ans. One, \$30; the other, \$40.*

19. If B were to give \$25 to A, they would have equal sums of money. If A were to give \$22 to B, he would then have twice as much as A. How much money has each?

20. A says to B, "Give me \$30 of your money, and my money will equal 4 times yours; but should I give you \$25 of my money, yours will equal 3 times mine." How much money has each?

21. A and B together perform a certain work in 30 days. At the end of 18 days B is called off, and A finishes it alone in 20 days. In what time could each perform the work alone?

SOLUTION. Let x = the number of days in which A could perform it alone, and y = the number of days in which B could perform it alone. Then, $\frac{1}{x}+\frac{1}{y}=\frac{1}{30}$, and $\frac{18}{x}+\frac{18}{y}+\frac{20}{x}=1$; whence, $x=50$ and $y=75$.

Here the process is shortened by multiplying the first equation by 18, and, without clearing of fractions, subtracting the result from the second equation.

22. A pound of tea and 5 pounds of sugar cost \$1.30, but if tea were to rise 50% and sugar 20%, the same quantities would cost \$1.77. Find the price of tea and sugar.

Ans. Tea, \$.70 per lb.; sugar, \$.12.

23. A workman having worked 12 days, and been idle 5, received \$25. When he worked 16 days, and was idle 7, he got \$33. What were his daily wages and his maintenance?

24. If A and B each gain \$24, A will have twice as much money as B, but if each lose \$24, A will have three times as much money as B. How much has each? *Ans. A, \$168; B, \$72.*

25. There are two kinds of tea—one worth \$1 per pound, the other worth \$1.25 per pound; how many pounds of each must be taken so that 112 pounds of the mixture may be worth \$130?

26. What are the two numbers whose sum is 133, and whose difference is 19?

27. On making up the roll of an army after a battle, it was found that the number of effective men was only 714 more than half the number before the battle. Of the remainder, the wounded were twice as many as the slain, the prisoners one-third as many as the slain, and the prisoners also one-third as many as were left for immediate service, while the number of wounded exceeded the number of prisoners by 677. What was the original strength of the army?

28. A sum of money was divided equally among a certain number of persons. If there had been six persons more, each would have received \$2 less than he did; and if there had been three persons less, each would have received \$2 more. Find the number of persons and how much each received.

SOLUTION.

$x =$ *the number of persons.*
$y =$ *the number of dollars each received.*
$xy =$ *the number of dollars divided.*

By the conditions,	$\begin{cases}(x+6)(y-2)=xy \\ (x-3)(y+2)=xy\end{cases}$	(1) (2)
Expanding (1),	$xy+6y-2x-12=xy$	
or,	$6y-2x=12$	(3)
Expanding (2),	$xy+2x-3y-6=xy$	
or,	$2x-3y=6$	(4)
Adding (3) *and* (4),	$3y=18$	
whence,	$y=6$	
Substituting value of y in (4),	$2x-18=6$	
whence,	$x=12$	

29. There is a certain rectangular floor, such that if it had been two feet broader and three feet longer, it would have been sixty-four square feet larger. But, if it had been three feet broader and two feet longer, it would have been sixty-eight square feet larger. Find the length and breadth of the floor.

30. A number of posts are placed at equal distances in a straight line. If to twice that number we add the number of the feet between two consecutive posts, the sum is 68. If from four times the number of the feet between two consecutive posts we subtract half the number of posts, the remainder is 68. Find the distance between the extreme posts.

Ans. $(24-1)20$, *or* 460, *feet.*

31. A certain company in a hotel found, when they came to pay their bill, that if there had been three more persons to pay the same bill, they would have paid one dollar each less than they did; and if there had been two fewer persons, they would have paid one dollar each more than they did. Find the number of persons and the number of dollars each paid.

SECTION XXVII.

SIMPLE EQUATIONS WITH MORE THAN TWO UNKNOWN QUANTITIES.

225. **Equations** with more than two unknown quantities can be solved by the methods of elimination given for solving equations with two unknown quantities.

226.—Ex. 1. Given $\left\{\begin{array}{l} 3x+2y-4z=8 \\ 5x-3y+3z=33 \\ 7x+y+5z=65 \end{array}\right\}$, to find x, y and z.

$$\begin{array}{lrr}
\textit{Three equations with three unknown quantities,} \left\{\begin{array}{l} \\ \\ \\ \end{array}\right. & \begin{array}{r} 3x+2y-4z=8 \\ 5x-3y+3z=33 \\ 7x+y+5z=65 \end{array} & \begin{array}{r} (1) \\ (2) \\ (3) \end{array} \\
 & \begin{array}{r} 9x+6y-12z=24 \\ 10x-6y+6z=66 \\ 14x+2y+10z=130 \end{array} & \begin{array}{r} (4) \\ (5) \\ (6) \end{array} \\
\textit{Two equations with two unknown quantities,} \left\{\begin{array}{l} \\ \\ \end{array}\right. & \begin{array}{r} 19x-6z=90 \\ 11x+14z=122 \end{array} & \begin{array}{r} (7) \\ (8) \end{array} \\
 & \begin{array}{r} 133x-42z=630 \\ 33x+42z=366 \end{array} & \begin{array}{r} (9) \\ (10) \end{array} \\
\textit{One equation with one unknown quantity,} \left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. & \begin{array}{r} 166x=996 \\ x=6 \\ z=4 \\ y=3 \end{array} & \begin{array}{r} (11) \\ (12) \\ (13) \\ (14) \end{array}
\end{array}$$

SOLUTION. From the three given equations obtain two equations, which shall not contain some one of the unknown quantities, say y. Multiplying equation (1) by 3 and equation (2) by 2, we obtain (4) and (5), in which the coefficients of y are the same. Multiply (3) by 2, making the coefficient of y the same as in (1). Adding (4) and (5), we have (7), and subtracting (1) from (6), we have (8). We now have two equations with but two unknown quantities.

We are now to obtain one equation with but one unknown quantity. Let us eliminate z. Multiplying equation (7) by 7, and (8) by 3, we obtain (9) and (10). Adding equations (9) and (10), we obtain (11). Dividing (11) by 166, we obtain (12), or $x=6$. Substituting this value of x in (8), and reducing, we get (13), or $z=4$. Substituting the value of x and z in (3), we obtain (14), or $y=3$.

2. Given $\left\{\begin{array}{l} x+y-z=3 \\ x+z-y=5 \\ y+z-x=7 \end{array}\right\}$, to find x, y and z.

SOLUTION. Adding equations (1) and (2), we obtain (4); whence, $x=4$. Adding (1) and (3), we obtain (5); hence, $y=5$. Adding (2) and (3), we obtain (6); whence, $z=6$.

$$\begin{array}{rr} x+y-z=3 & (1) \\ x+z-y=5 & (2) \\ \underline{y+z-x=7} & (3) \\ 2x=8 & (4) \\ x=4 & \\ 2y=10 & (5) \\ y=5 & \\ 2z=12 & (6) \\ z=6 & \end{array}$$

3. Given $\left\{\begin{array}{l} x+2y-3z=-\ 4 \\ x+2z-\ y=\ \ 9 \\ 2y+2z-\ x=\ 15 \end{array}\right\}$, to find x, y and z.

Ans. $x=3$; $y=4$; $z=5$.

4. Given $\left\{\begin{array}{l} y+z=\ 7 \\ x+z=\ 9 \\ x+y=10 \end{array}\right\}$, to find x, y and z.

5. Given $\left\{\begin{array}{l} 2x+\ 7y-11z=\ \ 10 \\ 5x-10y+\ 3z=-15 \\ -6x+12y-\ \ z=\ \ 31 \end{array}\right\}$, to find x, y and z.

Ans. $x=8$; $y=7$; $z=5$.

6. Given $\left\{\begin{array}{l} x+\ y+\ z=29 \\ x+2y+3z=62 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10 \end{array}\right\}$, to find x, y and z.

7. Given $\left\{\begin{array}{l} x+\frac{y}{2}+\frac{z}{3}=32 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=15 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=12 \end{array}\right\}$, to find x, y and z.

8. Given $\left\{\begin{array}{l} \frac{x}{a}+\frac{y}{b}=3 \\ \frac{y}{b}+\frac{z}{c}=5 \\ \frac{x}{a}+\frac{z}{c}=4 \end{array}\right\}$, to find x, y and z.

Ans. $x=a$; $y=2b$; $z=3c$.

9. Given $\left\{\begin{aligned} 2x-3y+2z&=13\\ 4y+2z&=14\\ 4u-2z&=26\\ 5y+3u&=32 \end{aligned}\right\}$, to find x, y, z and u.

SECTION XXVIII.

PROBLEMS PRODUCING SIMPLE EQUATIONS WITH MORE THAN TWO UNKNOWN QUANTITIES.

227.—Ex. 1. Find three numbers, A, B and C, such that A added to half of B, B added to a third of C, and C added to a fourth of A may each be 1000. *Ans. A, 640; B, 720; C, 840.*

2. Divide \$200 among three persons, A, B and C, so that twice A's share +\$80, three times B's share +\$30, and four times C's share +\$40, may each equal the same sum.

3. A man, talking with his wife and son of their ages, said that his age added to that of his son was 16 years more than that of his wife; the wife said that her age added to that of her son made 8 years more than that of her husband, and that all their ages together amounted to 88 years. What was the age of each? *Ans. Husband, 40; wife, 36; son, 12.*

4. A and B working together can earn \$40 in 6 days, A and C together can earn \$54 in 9 days, and B and C together can earn \$80 in 15 days. Find what each man alone can earn per day.

5. At an election there were two members of the legislature to be chosen, and there were three candidates, Lawton, Bowen and Stone. Lawton obtained 1056 votes, Bowen 987, and Stone 933. Now, 85 voted for Bowen and Stone, 744 for Bowen only, and 98 for Stone only. How many voted for Lawton and Stone? How many for Lawton and Bowen? How many for Lawton only? *Ans. For Lawton only, 148; for Lawton and Stone, 750; for Lawton and Bowen, 158.*

6. A's money, together with twice that of B and C, amounts to \$1050; B's, together with thrice that of A and C, amounts to \$1400; and C's, together with four times that of A and B, amounts to \$1650. How much money has each?

7. A man bought 10 horses, 120 cows and 46 colts. The price of 3 cows is equal to that of 5 colts. A horse, a cow and a colt together cost a number of dollars greater by 300 than the whole number of animals bought; and the whole sum expended was \$9366. Find the price of a horse, a cow and a colt, respectively.

Ans. A horse, \$420; a cow, \$35; a colt, \$21.

8. A, B and C can do a piece of work in 15 days; A and B together do four-thirds of what C does; and C does twice as much as A. Find the time in which each alone could do the work.

SECTION XXIX.

REVIEW PROBLEMS.

228.—Ex. 1. Given $\left\{\begin{array}{l} 6x+4y=56 \\ 4x-3y=\ \ 9 \end{array}\right\}$, to find x and y.

Ans. $x=6$; $y=5$.

2. Given $\left\{\begin{array}{l} 7x-3y=54 \\ 6x-5y=39 \end{array}\right\}$, to find x and y.

3. Given $\left\{\begin{array}{l} \dfrac{2x}{3}+5y=\ \ 23 \\ 5x+\dfrac{7y}{4}=-6\frac{1}{4} \end{array}\right\}$, to find x and y.

4. A gentleman divided \$1.20 among three poor persons; to the second he gave twice, and to the third three times, as much as to the first. How much did he give to each?

5. If a certain number be multiplied by m and by n, the sum of the products will be a. What is the number?

SOLUTION.

$$x = \textit{the number}.$$
$$mx + nx = a$$
$$\textit{Factoring, } (m+n)x = a$$
$$\textit{Dividing by } (m+n),\ x = \frac{a}{m+n}$$

6. A man bought an equal number of tons of coal for \$5 and for \$7 a ton, and the cost of the whole was \$492. How many tons of each did he buy? *Ans.* *41.*

7. A owes \$1200 and B \$2500, but neither has money enough to pay his debts. "Lend me," said A to B, "the eighth part of your money, and I shall be enabled to pay my debts." B answered, "I can discharge my debts if you will lend me the ninth part of yours." How much money had each?

8. What fraction is that to the numerator of which if 1 be added, the value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$? *Ans.* $\frac{4}{15}$.

9. Two persons, A and B, have the same income. A saves one-fifth of his yearly, but B, by spending \$50 per annum more than A, at the end of four years finds himself \$100 in debt. What is their income?

10. The grading of a square piece of land at \$2 a square yard cost as much as the inclosing of it at \$5 a linear yard. Required the side of the square.

SOLUTION.

$$x = \textit{the number of yards in the side of the square.}$$
$$4x = \textit{the number of yards around the square.}$$
$$x^2 = \textit{the number of square yards of grading.}$$
$$\$5 \times 4x = \$20x = \textit{price of inclosing.}$$
$$\$2 \times x^2 = \$2x^2 = \textit{price of grading.}$$
$$\$2x^2 = \$20x$$
$$\textit{Dividing by } \$2x,\quad x = 10$$

11. A general, ranging his army in the form of a solid square, finds he has 284 men to spare; but increasing the side by one man, he wants 25 to fill up the square. How many soldiers has he? *Ans.* *24000.*

12. A company of 180 persons consists of men, women and children. The men are 8 more in number than the women, and the children 20 more than the men and women together. How many of each are in the company?

Ans. *44 men ; 36 women ; 100 children.*

13. Given $4x+3y-z=31$, $3x-y-2z=0$ and $6x+2y+3z=45$, to find x, y and z.

14. It is required to divide 90 into four such parts that if the first part be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product and quotient shall all be equal.

15. There are three numbers such that their sums, taken two by two, are 11, 12 and 13, respectively. What are the numbers?

Ans. *5, 6 and 7.*

Test Questions.

229.—1. What is the degree of an *Equation?* What is a simple equation? What is the rule for clearing a simple equation of fractions?

2. How may a simple equation be solved? What is the root of an equation? How is the root verified? What is the rule for solving a simple equation with one unknown quantity?

3. What are *Simultaneous Equations?* What are independent equations?

4. What is *Elimination?* Elimination by comparison? Elimination by substitution? Elimination by addition or subtraction?

5. What is the *Rule* for elimination by comparison? For elimination by substitution? For elimination by addition or subtraction? How may equations with more than two unknown quantities be solved?

SECTION XXX.

INVOLUTION.

230.—Ex. 1. What is the product of a by a? What power of a does a^2 represent?

2. What is the product of a^2 by a? What power of a does a^3 represent?

3. How many times is a taken as a factor in producing a^2? In producing a^3?

4. Of what factors is a^2 the product? Of what factors is ab the product?

5. Is a^3 the product of equal or unequal factors? Is 6 the product of equal or unequal factors?

6. How will you find the third power of 2? Of x?

7. What is the fourth power of a? The second power of $-a$? The third power of $-a$?

Definitions.

231. A **Perfect Power** is the product of equal factors.

Thus, a^2, b^3, 4, 9, etc. are perfect powers.

232. An **Imperfect Power** is the product of unequal factors.

Thus, ab, $cd+xn$, 6, 15, etc. are imperfect powers.

233. **Involution** is the process of finding the powers of a quantity.

It is performed by successive multiplications, the quantity being taken as many times as there are ones in the index of the power.

Thus, a is involved to the third power by taking a three times as a factor.

234. Principles.—1. *All powers of a positive quantity are positive.*

For, taking a positive quantity any number of times as a factor must give a positive result. Thus, $a \times a = a^2$; $a^2 \times a = a^3$.

2. *All the even powers of a negative quantity are positive, and all the odd powers are negative.*

For, taking a negative quantity twice as a factor, or multiplying it by itself, gives a positive result; multiplying that result by the quantity which is negative, gives for the third power a negative result. Thus, $(-a) \times (-a) = a^2$; $a^2 \times (-a) = -a^3$; $(-a^3) \times (-a) = a^4$, etc.

CASE I.

Involution of Monomials.

235.—Ex. 1. Find the cube or third power of $5a^2x$.

SOLUTION. $5a^2x$ taken three times as a factor is equal to $5a^2x \times 5a^2x \times 5a^2x$, or to $5 \times 5 \times 5a^2a^2a^2xxx$, whose product is $125a^6x^3$.

$$\begin{aligned}(5a^2x)^3 &= 5a^2x \times 5a^2x \times 5a^2x \\ &= 5 \times 5 \times 5a^2a^2a^2xxx \\ &= 125a^6x^3\end{aligned}$$

2. Find the square of $-3ax$. *Ans.* $9a^2x^2$.

236. Rule for Involution of Monomials.—*Involve the numerical coefficient to the required power, multiply the exponent of each letter by the number denoting the exponent of the required power, and give the proper sign to the result.*

A fraction is involved by involving both numerator and denominator.

PROBLEMS.

1. Find the cube of $7ab$. *Ans.* $343a^3b^3$.
2. Find the square of $6a^2$.
3. Find the third power of $-a^2b^3$. *Ans.* $-a^6b^9$.
4. Find the fourth power of ab^2c^3.
5. Find the fifth power of $-a^2b^3c^4$. *Ans.* $-a^{10}b^{15}c^{20}$.
6. Find the mth power of abc^2.
7. Find the third power of $x^{-n}y^{-m}$. *Ans.* $x^{-3n}y^{-3m}$.

8. Find the fifth power of $-3ax^5$.

9. What is the nth power of x^2y^3?

10. What is the nth power of $-3x^2y^3$? *Ans.* $\pm 3^n x^{2n} y^{3n}$.

Here, n may be any number whatever; hence, the nth power of the given quantity may be even or odd, and consequently may be either positive or negative, as is indicated by the sign $\pm$. When a certain number is given as the value of n, the power is positive if the number is even, and negative if odd.

11. What is the second power of $\dfrac{6a^2}{c}$?

SOLUTION.

$$\left(\frac{6a^2}{c}\right)^2=\frac{6a^2}{c}\times\frac{6a^2}{c}=\frac{36a^4}{c^2}.$$

12. What is the cube of $\dfrac{3x}{4a}$? *Ans.* $\dfrac{27x^3}{64a^3}$.

13. What is the fifth power of $\dfrac{4a^2}{5x}$?

14. What is the mth power of $\dfrac{a^{-n}c^2}{a^{-n}b^{3-n}}$? *Ans.* $\dfrac{a^{-mn}c^{2m}}{a^{-mn}b^{3m-mn}}$.

15. What is the value of $(-4ab^2c^3m^2)^4$?

16. What is the value of $(6a^2b^{-2}c^{-3}x)^3$? *Ans.* $216a^6b^{-6}c^{-9}x^3$.

CASE II.

Involution of Polynomials.

237.—Ex. 1. Find the square and the cube of $(a+b)$.

SOLUTION. The square of $a+b$, or $(a+b)(a+b)$, is, by actual multiplication, $a^2+2ab+b^2$, and the cube of $a+b$, or $(a+b)(a+b)(a+b)$, is, by actual multiplication, $a^3+3a^2b+3ab^2+b^3$.

$$\begin{array}{l}
a+b\\
\underline{a+b}\\
a^2+ab\\
\underline{\quad +ab+b^2}\\
a^2+2ab+b^2, \textit{ square of } (a+b).\\
\underline{a+b}\\
a^3+2a^2b+ab^2\\
\underline{\quad +a^2b+2ab^2+b^3}\\
a^3+3a^2b+3ab^2+b^3, \textit{ cube of } (a+b).
\end{array}$$

2. Find the cube of $(a-b)$. *Ans.* $a^3-3a^2b+3ab^2-b^3$.

238. Rule for Involution of Polynomials.—*Use the given quantity as a factor as many times as are indicated by the exponent of the required power.*

PROBLEMS.

1. Find the square of $(a+c)$. *Ans.* $a^2+2ac+c^2$.

2. Find the square of $(a+2x)$.

3. Find the square of $(a-c)$. *Ans.* $a^2-2ac+c^2$.

4. Find the square of $(2a+c)$.

5. Find the square of $(ac-b)$. *Ans.* $a^2c^2-2abc+b^2$.

6. What is the cube of $(m+n)$?

7. What is the cube of $(2a+3b)$?
Ans. $8a^3+36a^2b+54ab^2+27b^3$.

8. What is the cube of $(2+x)$?

9. What is the value of $(3-2x)^3$?
Ans. $27-54x+36x^2-8x^3$.

10. Expand $(x-2)^4$.

11. Expand $(a+b+c)^2$. *Ans.* $a^2+2ab+2ac+b^2+2bc+c^2$.

12. Involve $1+x-x^2$ to the third power.

239. Any **Polynomial** may be squared without recourse to multiplication by observing the followiug case and the principles drawn therefrom.

Let $a+b+c+d$ be any polynomial; then by actual multiplication,

$$(a+b+c+d)^2=a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2.$$

Changing the order of terms, we have

$$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2bc+2ad+2bd+2cd.$$

Factoring and changing the order, we have

$$(a+b+c+d)^2=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2.$$

240. Principles.—1. *The square of any polynomial consists of the square of each term, together with twice the product of every pair of terms; or,*

2. *The square of any polynomial consists of the square of the first term, plus twice the product of the first term into the second, plus the square of the second, plus twice the sum of the first two terms into the third, plus the square of the third, etc.*

PROBLEMS.

1. Find the square of $a-b+c$.

Ans. $a^2+b^2+c^2-2ab+2ac-2bc$.

2. Find the square of $x+y+z$.

3. What is the second power of $a+m-n$?

Ans. $a^2+m^2+n^2+2am-2an-2mn$.

4. Expand $(2+3x+4x^2)^2$.

5. Expand $(2a-b+c-d)^2$.

Ans. $4a^2+b^2+c^2+d^2-4ab+4ac-4ad-2bc+2bd-2cd$.

SECTION XXXI.

EVOLUTION.

241.—Ex. 1. What is the product of $b\times b$? Of $x\times x\times x$? What is one of the equal factors of b^2? Of x^3?

2. What is the second or square root of 9? The third or cube root of 27?

3. What is the square root of a^2? What is the cube root of a^3? Of $-a^3$?

Definitions.

242. Evolution is the process of finding a root of a quantity.

It consists in determining the quantity which, when involved to the power corresponding to the required root, will equal the given quantity.

This process is sometimes called *extraction of the root.*

243. In **Fractional Exponents** the numerator denotes a *power*, and the denominator a *root.*

Thus, $a^{\frac{2}{3}}$, or $\sqrt[3]{a^2}$, denotes the cube root of a square.

244. Principles.—1. *An odd root of a quantity has the same sign as the quantity.*

Thus, the cube root of a^3 is a, and the cube root $-a^3$ is $-a$.

2. *An even root of a positive quantity is either positive or negative.*

Thus, $a \times a = a^2$, and $-a \times -a = a^2$; hence, the square root of a^2 is either a or $-a$; that is, it is either positive or negative.

3. *An even root of a negative quantity is impossible.*

Thus, there is no square root of $-a^2$, for if any quantity be multiplied by itself, the result is a positive quantity. Hence, an indicated even root of a negative quantity is called an *impossible* or an *imaginary* quantity.

CASE I.

Evolution of Monomials.

245.—Ex. 1. Find the cube root of $125a^6x^3$.

$$\sqrt[3]{125a^6x^3} = \sqrt[3]{125} \times a^{\frac{6}{3}}x^{\frac{3}{3}} = 5a^2x$$

Solution. To cube a monomial we cube the numerical coefficient, and multiply the exponents of the letters by 3; hence, to find the cube root of the given monomial we reverse the process; that is, we extract the cube root of its coefficient, and divide the exponents of the letters by 3. The cube root thus found is $5a^2x$.

2. Find the square root of $36a^2b^4$.

SOLUTION. Since finding the square root of a quantity is the reverse of squaring the root, we extract the square root of the coefficient of the given quantity, and divide the exponents of its letters by 2. The square root thus found, which may be either positive or negative (Art. 244, 2), is $\pm 6ab^2$.

$$\sqrt{36a^2b^4} = \sqrt{36} \times a^{\frac{2}{2}}b^{\frac{4}{2}} = \pm 6ab^2$$

3. Find the fourth root of $81x^8y^4$. *Ans.* $\pm 3x^2y$.

246. Rule for Evolution of Monomials.—*Extract the required root of the numerical coefficient, divide the exponent of each letter by the number denoting the index of the required root, and give the proper sign to the result.*

The root of a fraction is found by taking the root of the numerator and of the denominator.

PROBLEMS.

1. What is the cube root of $-8x^3y^6$? *Ans.* $-2xy^2$.

2. What is the fourth root of $625a^4y^8$?

3. What is the cube root of $-2^3a^6b^9c^{12}$? *Ans.* $-2a^2b^3c^4$.

4. Find the cube root of $-64a^6y^3$. *Ans.* $-4a^2y$.

5. Find the mth root of $7^ma^{3m}x^{2m}$.

6. Find the fifth root of a^2x^{10}. *Ans.* $a^{\frac{2}{5}}x^2$.

7. What is the value of $\sqrt[4]{625x^4y^4z^{12}}$?

8. What is the value of $\sqrt[3]{-8a^{-3}b^6x^{-2}}$? *Ans.* $-2a^{-1}b^2x^{-\frac{2}{3}}$.

9. What is the square root of $\dfrac{16a^2}{9b^2}$?

SOLUTION.

$$\sqrt{\frac{16a^2}{9b^2}} = \frac{\sqrt{16a^2}}{\sqrt{9b^2}} = \pm\frac{4a}{3b}.$$

10. What is the fourth root of $\dfrac{a^4b^8c^{12}}{16x^4y^8}$?

11. What is the third root of $-\dfrac{8a^3y^6}{27x^9}$? *Ans.* $-\dfrac{2ay^2}{3x^3}$.

12. What is the fifth root of $\dfrac{32a^5b^{10}}{c^{15}}$?

CASE II.

Square Root of Polynomials.

247.—Ex. 1. Find the square root of $a^2+2ab+b^2$.

$$
\begin{array}{lll}
 & a^2+2ab+b^2 & (a+b \\
 & a^2 & \\ \hline
2a &)\ 2ab+b^2 & \\
(2a+b)b & = 2ab+b^2 & \\ \hline
\end{array}
$$

Solution. From the nature of the square of a polynomial (Art. 239, Prin. 2), the first term of the given square will be the square of the first term of the required root. The first term of the square is a^2, and its square root is a, which we write as the first term of the root. Subtracting a^2 from the whole expression, we have left $2ab+b^2$, which must be two times the product of the two terms of the root plus the square of the last term of the root (Art. 239, Prin. 2). Dividing $2ab$, the first term of the dividend, by $2a$, which is double the first term of the root, we obtain b, the other, or second, term of the root. Adding the b to the trial divisor $2a$, we obtain the complete divisor $2a+b$. Multiplying this by b, the second term, and subtracting, we have no remainder. Hence, $a+b$ is the root required.

2. Find the square root of $a^2+2ab+b^2-2ac-2bc+c^2$.

$$
\begin{array}{lll}
 & a^2+2ab+b^2-2ac-2bc+c^2 & (a+b-c \\
 & a^2 & \\ \hline
2a &)\ 2ab+b^2 & \\
(2a+b)b & = 2ab+b^2 & \\ \hline
2a+2b &)\ -2ac-2bc+c^2 & \\
(2a+2b-c)c & = -2ac-2bc+c^2 & \\ \hline
\end{array}
$$

Solution. We find the first two terms of the root, $a+b$, as before. Dividing the remainder by twice this root, we obtain $-c$, the third term of the root. Adding the $-c$ to the trial divisor, we obtain the complete divisor, $2a+2b-c$. Multiplying this by $-c$, the third term of the root, and subtracting, we have no remainder. Hence, $a+b-c$ is the root required.

3. Find the square root of $4x^2+12xy+9y^2$. *Ans.* $2x+3y$.

248. Rule for Extracting the Square Root of Polynomials.— *Arrange the terms according to the powers of some letter.*

Find the greatest square root of the first term, and place it at the right for the first term of the required root, and subtract its square from the given polynomial.

Divide the first term of the remainder by double the root already found, and annex the result to the root, and also to the divisor, for a complete divisor.

Multiply the complete divisor by the second term of the root, and subtract the product from the dividend.

Continue the process, if there are other terms, as before.

PROBLEMS.

1. Find the square root of $9x^4-24x^2+16$. *Ans.* $3x^2-4$.

2. Find the square root of $16x^4-8x^2cy+c^2y^2$.

3. Find the square root of $36x^6+12x^3+1$. *Ans.* $6x^3+1$.

4. Find the square root of $1-2x+5x^2-4x^3+4x^4$.

5. Find the square root of x^4-4x^3+8x+4. *Ans.* x^2-2x-2.

6. What is the value of $\sqrt{4a^2+4ax+x^2-4ay-2xy+y^2}$?

7. What is the value of $(9x^2+30xy+25y^2)^{\frac{1}{2}}$?

8. What is the value of $\sqrt{a^2-2ab+b^2+2ac-2bc+c^2}$?

249. Any **Trinomial** that is a perfect square has two of its terms squares, and the other term is double the product of their square roots. Hence, to obtain the square root of a trinomial which is a perfect square,—

Arrange the trinomial according to the powers of either of its letters, if necessary; extract the square roots of the two extreme terms, and unite their roots by the sign of the middle term.

PROBLEMS.

1. What is the square root of $16a^2-56ab+49b^2$? *Ans.* $4a-7b$.

2. What is the square root of $64a^2+48abc+9b^2c^2$?

3. What is the square root of $4m^2+12mn+9n^2$?

CASE III.

Square Root of Numbers.

250. A **Method** of extracting the square root of numbers may be derived from the nature of algebraic polynomial squares.

251. Principles.—1. *If a square expressed by more than two orders of figures be separated into periods of two figures each, commencing with the units, the number of figures in the root is indicated, and also the orders of the squares in which are to be found the squares of the numbers in the orders of the root.*

For the square of 1 is 1, the square of 10 is $\dot{1}0\dot{0}$, the square of 100 is $\dot{1}0\dot{0}0\dot{0}$, etc.

2. *The square of any number expressed by more than one order of figures consists of the square of the tens, plus twice the product of the tens by the units, plus the square of the units.*

For any number expressed by more than one order of figures may be regarded as an algebraic polynomial, and as composed of two parts, units and tens.

Thus, if t denote *tens* and u denote units, any number will be denoted by $t+u$, and

$$(t+u)^2=t^2+2tu+u^2=t^2+(2t+u)u.$$

Then, let $t=3$ and $u=7$, and $t+u=37$; hence,

$$37^2=30^2+2\times30\times7+7^2=1369.$$

252.—Ex. 1. What is the square root of 2209?

Solution. Since the given number is expressed by 4 orders of figures, it may be separated into two periods, and its square root will consist of 2 figures, and will express tens and units.

$$
\begin{array}{lrll}
 & \dot{2}2\dot{0}9 = & t^2+2tu+u^2 & (\overset{t}{4}\,\overset{u}{7} \\
 & 4^2 = 16 = & t^2 & \\ \hline
2t = 2\times 4 \text{ tens} = 80 &)\,609 = & 2tu+u^2 = (2t+u)u & \\
u = 7 & & & \\ \hline
2t+u = 87 & & & \\
87\times 7 = & 609 = & (2t+u)u & \\ \hline
 & 0 & 0 &
\end{array}
$$

Representing the tens by t and the units by u, we have $2209 = t^2+2tu+u^2$.

Then t^2, or the greatest square of tens which is less than 2200, is 16 hundreds, whose root is 4 tens. Subtracting 16 hundreds from the given number, and its equal t^2 from the expression above, 609 remains, which is equal to $2tu+u^2$, or $(2t+u)u$. Dividing this remainder by $2t$, that is, by 80, we obtain 7, the value of u. Then $(2t+u)u$, that is, 87×7, is equal to 609. Subtracting this product and its equal $(2t+u)u$, from the quantities above them, there is no remainder. Hence, the square root of 2209 is 47.

2. What is the square root of 55225?

Solution.

$$
\begin{array}{lrll}
 & \dot{5}5\dot{2}|\dot{2}5 = & t^2+2tu+u^2 & (\overset{t}{2}\,\overset{u}{3}\,\overset{u}{5} \\
 & 2^2 = 4 = & t^2 & \\ \hline
2t = 2\times 2 \text{ tens} = 40 &)\,152 = & 2tu+u^2 = 2(t+u)u & \\
u = 3 & & & \\ \hline
2t+u = 43 & & & \\
43\times 3 = & 129 = & (2t+u)u & \\ \hline
2t = 2\times 23 \text{ tens} = 460 &)\,2325 = & 2tu+u^2 = (2t+u)u & \\
u = 5 & & & \\ \hline
2t+u = 465 & & & \\
465\times 5 = & 2325 = & (2t+u)u & \\ \hline
 & 0 & 0 &
\end{array}
$$

In the same manner as in the preceding solution, we find 23, the tens and units of the tens of the root of the greatest square of tens which is contained in 55200, and have the remainder 2325.

We now consider 55225 as the square of 23 tens and some number of units of units. Representing the 23 tens of the root, which have been

found, by t, and the units of the root to be found, by u, we may represent the square of the root by $t^2+2tu+u^2$. The equal of t^2, or 230^2, has been subtracted; hence, the remainder, 2325, is equal to $2tu+u^2$, or $(2t+u)u$.

Dividing the remainder by $2t$, that is, by 460, we obtain 5, the value of u. Then $(2t+u)u$, that is, 465×5, is equal to 2325. Subtracting this product and its equal, $(2t+u)u$, from the quantities above them, we have no remainder. Hence, the square root of 55225 is 235.

If the given number had been a decimal fraction, its periods would have been pointed off to the right, beginning with units; for the square of .1 is .01, the square of .09 is .0081, etc.

3. Find the square root of 77841. *Ans. 279.*

253. Rule for the Extraction of the Square Root of Numbers.—*Point off the given number into periods of two orders each, beginning with the units and proceeding toward the left and right.*

Find the greatest square in the highest period, considered as units, and place its root at the right for the first figure of the required root. Subtract this square from the highest period, and to the remainder bring down the next period for a dividend.

Divide the dividend, omitting its right-hand order, by twice the root already found, and write the quotient for the second figure of the required root.

To the divisor add the part of the root found by it, multiply the result by that part of the root, and subtract the product from the dividend.

Continue the process, if there are other periods, as before.

When 0 occurs in the root, instead of indicating the multiplication by 0 and subtracting, it is simpler to annex a cipher to the divisor, and to the dividend bring down another period.

If there be a remainder after all the periods have been used, periods of decimals may be formed by annexing ciphers, and the work continued.

When the given quantity is a fraction whose numerator and denominator are not both squares, reduce the fraction to a decimal before extracting the root.

PROBLEMS.

1. Find the square root of 177241. *Ans. 421.*

2. Find the square root of 3249.

3. Find the square root of 165649. *Ans. 407.*

4. Find the square root of 41.2164.

5. Find the square root of 2.5. *Ans. 1.5811.*

6. Find the square root of 17.3056.

7. What is the square root of .001849? *Ans. .043.*

8. What is the square root of 2 to two decimal orders?

9. What is the square root of $\frac{2916}{3481}$? *Ans.* $\frac{54}{59}$

10. What is the square root of $\frac{1}{26}$ to three decimal orders?

11. What is the value of $\sqrt{2\frac{47}{121}}$? *Ans.* $1\frac{6}{11}$.

12. What is the value of $\sqrt{\frac{5}{11}}$ to three decimal orders?

13. What is the square root of .008723 to three orders?

14. What is the square root of $6\frac{4}{7}$ to three decimal orders?

SECTION XXXII.

RADICALS.

254.—Ex. 1. What is expressed by $\sqrt{4a^2}$? By $9a^{\frac{1}{3}}$?

2. In what two ways may the root of a quantity be indicated?

3. In $3\sqrt{a}$, what shows how many times $\sqrt{a}$ is taken?

4. Name two different expressions having the same quantity under the same radical sign.

5. Can the root indicated by $\sqrt{4a^2}$ be exactly obtained? Can the root indicated by $\sqrt[3]{2a}$ be exactly obtained?

Definitions.

255. A **Radical** is an indicated root of a quantity.

Thus, $\sqrt{a}$, $64^{\frac{1}{3}}$, $\sqrt{2ax^5}$, etc., are radicals.

256. The **Coefficient** of a radical is the quantity prefixed to the radical to show how many times it is taken.

Thus, in $2\sqrt{ax^5}$, 2 is the coefficient of $\sqrt{ax^5}$.

257. The **Degree** of a radical is indicated by the index of its root, or by the denominator of its fractional exponent.

Thus, $\sqrt{ab}$, $x^{\frac{1}{2}}$, $(ax)^{\frac{1}{2}}$, are radicals of the second degree, and $\sqrt[3]{xy}$, $\sqrt[3]{m}$, $(3ab^2)^{\frac{1}{3}}$, are radicals of the third degree.

258. **Similar Radicals** are those which have the same quantity under the same radical sign.

Thus, $\sqrt{a^3b}$, $2\sqrt{a^3b}$, $(a^3b)^{\frac{1}{2}}$, are similar radicals.

259. A **Rational Quantity** is a quantity whose indicated root can be exactly obtained, and an **Irrational Quantity, or Surd,** is a quantity whose indicated root cannot be exactly obtained.

Thus, $\sqrt{9a^2}$ and $\sqrt[3]{27a^3}$ are rational quantities, and $\sqrt{3b}$ and $\sqrt[3]{11x^2}$ are irrational quantities.

260. Principles.—1. *The root of any quantity is equal to the product of the roots of its factors.*

Thus, $\sqrt{64}=\sqrt{4}\times\sqrt{16}=2\times 4=8$.

2. *The product of the same roots of two quantities is equal to the same root of their product.*

Thus, $\sqrt{4}\times\sqrt{16}=\sqrt{64}$.

3. *The quotient of the same roots of two quantities is equal to the same root of their quotient.*

Thus, $\sqrt{64}\div\sqrt{4}=\sqrt{16}$.

CASE I.

A Radical to its Simplest Form.

261. A Radical is in its **Simplest Form** when it contains no factor whose indicated root can be obtained.

262.—Ex. 1. Reduce $\sqrt{80a^3b^2}$ to its simplest form.

$$\sqrt{80a^3b^2}=\sqrt{16a^2b^2\times 5a}$$
$$=\sqrt{16a^2b^2}\times\sqrt{5a}=4ab\sqrt{5a}$$

Solution. We separate the quantity under the radical sign into two factors, $16a^2b^2$, which is a perfect square, and $5a$, which is a surd. Then, by Prin. 1, Art. 260, $\sqrt{80a^3b^2}=\sqrt{16a^2b^2}\times\sqrt{5a}$; hence, $\sqrt{80a^2b^2}$ is equal to the product of the square root of $16a^2b^2$ by $\sqrt{5a}$, or $4ab\sqrt{5a}$.

2. Reduce $5\sqrt[3]{24x^4}$ to its simplest form.

$$5\sqrt[3]{24x^4}=5\sqrt[3]{8x^3\times 3x}$$
$$=5\sqrt[3]{8x^3}\times\sqrt[3]{3x}$$
$$=5\times 2x\times\sqrt[3]{3x}$$
$$=10x\sqrt[3]{3x}$$

Solution. We separate the quantity under the radical sign into two factors, $8x^3$, which is a perfect cube, and $3x$, which is a surd. Multiplying the coefficient 5 by $2x$, the cube root of $8x^3$, and that product by the surd, $\sqrt[3]{3x}$, we have $10x\sqrt[3]{3x}$, the simplest form of the given quantity.

3. Reduce $\sqrt{27a^5b^3}$ to its simplest form. *Ans.* $3a^2b\sqrt{3ab}$.

263. Rule for Reducing a Radical to its Simplest Form.—*Separate the given radical into two radical factors, one of which is rational and the other a surd. Find the root of the rational factor, multiply it by the coefficient of the given radical, and place the product as the coefficient of the surd.*

PROBLEMS.

Reduce the following radicals to their simplest form.

1. $\sqrt{18x}$. *Ans.* $3\sqrt{2x}$.

2. $3\sqrt{x^3y^2z}$.

3. $\sqrt{150a^2b}$. *Ans.* $5a\sqrt{6b}$.

4. $\sqrt[3]{56a^4b^5}$.

5. $2\sqrt[3]{54a^4b^3c^2}$. *Ans.* $6ab\sqrt[3]{2ac^2}$.

6. $20x\sqrt{75x^3y}$.

7. $\sqrt[3]{ax^3+bx^6}$. *Ans.* $x\sqrt[3]{a+bx^3}$.

8. $5\sqrt{25a-25}$.

9. $ab(a^3b^2-a^2b^2)^{\frac{1}{2}}$. *Ans.* $a^2b^2\sqrt{a-1}$.

10. $7\sqrt{(a^2-b^2)(a+b)}$.

264. When the **Radical** is fractional, to have only an integral quantity under the radical sign,

Multiply both terms of the fraction by that quantity which will make its denominator a perfect power of the same degree as the root indicated.

Reduce to the simplest form—

SOLUTION.

1. $5\sqrt[3]{\frac{2}{3}}$. $5\sqrt[3]{\frac{2}{3}}=5\sqrt[3]{\frac{2\times 9}{3\times 9}}=5\sqrt[3]{\frac{1}{27}\times 18}=5\times\frac{1}{3}\sqrt[3]{18}=\frac{5}{3}\sqrt[3]{18}$.

2. $\sqrt{\frac{50}{147}}$. *Ans.* $\frac{5}{21}\sqrt{6}$.

3. $\left(\frac{7}{8a}\right)^{\frac{1}{2}}$.

4. $\sqrt{\frac{2a^2}{3}}$. *Ans.* $\frac{a}{3}\sqrt{6}$.

5. $\frac{1}{2}\sqrt{\frac{x}{15}}$.

6. $\frac{4}{7}\sqrt[3]{\frac{3}{16}}$. *Ans.* $\frac{1}{7}\sqrt[3]{12}$.

7. $10\sqrt{\frac{3}{50}}$. *Ans.* $\sqrt{6}$.

8. $6\sqrt{\frac{7}{8}}$.

9. $5\sqrt{\frac{9}{10}}$. *Ans.* $\frac{3}{2}\sqrt{10}$.

10. $4b\sqrt[3]{\frac{3a^4}{4b^2}}$.

11. $2\sqrt[3]{\frac{4a^4}{9}}$. *Ans.* $\frac{2a}{3}\sqrt[3]{12a}$.

CASE II.

A Rational Quantity to the Form of a Radical.

265.—Ex. 1. Reduce $5ab^2$ to the form of a cube root.

SOLUTION. Since any quantity is equal to the cube root of its cube, $5ab^2$ is equal to the cube root of $(5ab^2)^3$, or to $\sqrt[3]{125a^3b^6}$.

$$5ab^2=\sqrt[3]{(5ab^2)^3}$$
$$=\sqrt[3]{125a^3b^6}$$

2. Reduce $5xy$ to the form of the square root.

Ans. $\sqrt{25x^2y^2}$.

266. Rule for Reducing a Rational Quantity to the Form of a Radical.—*Involve the quantity to the power denoted by the given root, and place the result under the corresponding radical sign.*

PROBLEMS.

1. Reduce $2ac$ to the form of the square root. *Ans.* $\sqrt{4a^2c^2}$.

2. Reduce $3x^2y^{-1}$ to the form of the cube root.

Ans. $\sqrt[3]{27x^6y^{-3}}$.

3. Reduce $-3x$ to a radical of the third degree.

4. Reduce $\frac{\sqrt{a}}{2a}$ to a radical of the second degree. *Ans.* $\sqrt{\frac{1}{4a}}$

5. Reduce x^2+y to the form of the cube root.

267. The **Coefficient** of a radical can be placed under the radical sign by *involving the coefficient to the power denoted by the radical, and multiplying the quantity already under the radical sign by the result.*

PROBLEMS.

1. Place the coefficient of $2a\sqrt{5x}$ under the radical sign.

SOLUTION.

$$2a\sqrt{5x}=\sqrt{(2a)^2\times 5x}=\sqrt{20a^2x}$$

2. Place the coefficient of $4\sqrt{cd^2}$ under the radical sign.

3. Reduce $\frac{3}{4}\sqrt[3]{mn}$ to a radical without a coefficient.

Ans. $\sqrt[3]{\frac{27mn}{64}}$.

4. Reduce $3a\sqrt[n]{b^2c^3}$ to a radical without a coefficient.

5. Place the factor 2 of the coefficient of $2ab\sqrt[3]{3c^2x}$ under the radical sign. *Ans.* $ab\sqrt[3]{24c^2x}$.

6. Place the coefficient of $5a\sqrt{3a^2x}$ under the radical sign.

CASE III.

Radicals to the same Degree.

268.—Ex. 1. Reduce $\sqrt{a}$ and $\sqrt[3]{b}$ to radicals of the same degree.

SOLUTION. Substituting for the radical signs fractional exponents, we have $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$, and reducing the exponents to equivalent ones having a common denominator, we have $a^{\frac{3}{6}}$ and $b^{\frac{2}{6}}$, or $\sqrt[6]{a^3}$ and $\sqrt[6]{b^2}$, which, having a common index, are of the same degree.

$$\sqrt{a}=a^{\frac{1}{2}}=a^{\frac{3}{6}}, \text{ or } \sqrt[6]{a^3}$$
$$\sqrt[3]{b}=b^{\frac{1}{3}}=b^{\frac{2}{6}}, \text{ or } \sqrt[6]{b^2}$$

2. Reduce $\sqrt{3}$ and $\sqrt[4]{4}$ to a common index.

Ans. $\sqrt[4]{9}$; $\sqrt[4]{4}$.

269. Rule for reducing Radicals to the same Degree.—*Reduce the given or equivalent fractional exponents to a common denominator. Involve each quantity to the power denoted by the numerator of the reduced exponent, and indicate the root denoted by the common denominator.*

PROBLEMS.

1. Reduce $\sqrt{5}$ and $\sqrt[3]{4}$ to radicals of the same degree.

Ans. $\sqrt[6]{125}$; $\sqrt[6]{16}$.

2. Reduce a^2 and $a^{\frac{1}{2}}$ to radicals of the same degree.

3. Reduce $\sqrt{a}$, $\sqrt[3]{a}$ and $\sqrt[4]{a}$ to radicals of the same degree.

Ans. $\sqrt[12]{a^6}$; $\sqrt[12]{a^4}$; $\sqrt[12]{a^3}$.

4. Reduce $\sqrt[n]{a}$ and $\sqrt[m]{b}$ to radicals of the same degree.

5. Reduce $\sqrt{\frac{1}{3}}$, $\sqrt[3]{\frac{3}{4}}$ and $\sqrt[4]{2}$ to radicals of the same degree.

Ans. $\frac{1}{3}\sqrt[12]{729}$; $\frac{1}{2}\sqrt[12]{1296}$; $\sqrt[12]{8}$.

6. Reduce $\sqrt{\frac{a}{2}}$ and $\sqrt[3]{\frac{a}{2}}$ to radicals with a common index.

CASE IV.

Addition of Radicals.

270.—Ex. 1. Add $4\sqrt{7}$ and $5\sqrt{7}$.

SOLUTION. 4 times $\sqrt{7}$ and 5 times $\sqrt{7}$ are (4 plus 5) times $\sqrt{7}$, or 9 times $\sqrt{7}$.

$$\begin{array}{r} 4\sqrt{7} \\ 5\sqrt{7} \\ \hline 9\sqrt{7} \end{array}$$

2. Add $\sqrt{8}$ and $\sqrt{50}$.

SOLUTION. Reducing the given radicals to their simplest form, we have $2\sqrt{2}$ and $5\sqrt{2}$. Then 2 times $\sqrt{2}$ and 5 times $\sqrt{2}$ are 7 times $\sqrt{2}$.

$$\begin{array}{r} \sqrt{8}=2\sqrt{2} \\ \sqrt{50}=5\sqrt{2} \\ \hline 7\sqrt{2} \end{array}$$

3. Add $\sqrt{72}$ and $\sqrt{128}$. *Ans.* $14\sqrt{2}$.

271. Rule for Addition of Radicals.—*Reduce the radicals, if necessary, to their simplest forms. If the radicals are then similar, add their coefficients, and annex to the sum the common radical; but if they are not similar, indicate their sum by the proper sign.*

PROBLEMS.

1. Add $\sqrt{12}$, $\sqrt{27}$ and $\sqrt{48}$. *Ans.* $9\sqrt{3}$.

2. Add $\sqrt{12a^2}$ and $\sqrt{27a^2}$.

3. Add $\sqrt[3]{56}$, $\sqrt[3]{189}$ and $\sqrt[3]{448}$. *Ans.* $9\sqrt[3]{7}$.

4. What is the sum of $5\sqrt{98a}$ and $10\sqrt{2a}$?

5. What is the sum of $\sqrt[3]{54x^2}$ and $\sqrt[3]{128x^2}$? *Ans.* $7\sqrt[3]{2x^2}$.

6. What is the sum of $\sqrt[3]{b^4y}$ and $\sqrt[3]{by^4}$?

7. Add $\sqrt{24}$, $2\sqrt{72}$ and $a\sqrt{bx^2}$.
Ans. $2\sqrt{6}+12\sqrt{2}+ax\sqrt{b}$.

8. Add $\frac{1}{2}\sqrt{a^2b}$ and $\frac{1}{3}\sqrt{4bx^4}$.

CASE V.

Subtraction of Radicals.

272.—Ex. 1. From $\sqrt{75}$ take $\sqrt{12}$.

SOLUTION. Reducing the given radicals to their simplest form, we have $5\sqrt{3}$ and $2\sqrt{3}$. Then, 2 times $\sqrt{3}$ taken from 5 times $\sqrt{3}$ leaves 3 times $\sqrt{3}$, or $3\sqrt{3}$.	$\sqrt{75}=5\sqrt{3}$ $\sqrt{12}=\underline{2\sqrt{3}}$ $3\sqrt{3}$

2. From $\sqrt[3]{125a^6b^4}$ take $b\sqrt[3]{8a^6b}$. *Ans.* $3a^2b\sqrt[3]{b}$.

273. Rule for Subtraction of Radicals.—*Reduce the radicals, if necessary, to their simplest forms. If the radicals are then similar, subtract the coefficient of the subtrahend from the coefficient of the minuend, and annex to the difference the common radical; but if they are not similar, indicate their difference by the proper sign.*

PROBLEMS.

1. From $\sqrt{108}$ take $\sqrt{27}$. *Ans.* $3\sqrt{3}$.

2. From $\sqrt{320}$ take $\sqrt{80}$.

3. From $\sqrt{448}$ take $2\sqrt{63}$. *Ans.* $2\sqrt{7}$.

4. From $3\sqrt{\frac{3}{4}}$ take $\sqrt{\frac{1}{3}}$.

5. From $9\sqrt[3]{5x^4}$ take $\sqrt[3]{135x^4}$. *Ans.* $6x\sqrt[3]{5x}$.

6. From $\sqrt{a^2x}$ take $\sqrt{c^2x}$.

7. From $2\sqrt{ax}$ take $\sqrt[3]{ax^2}$. *Ans.* $2\sqrt{ax}-\sqrt[3]{ax^2}$.

8. From $8\sqrt[3]{a^3b}$ take $2\sqrt[3]{a^6b}$.

CASE VI.

Multiplication of Radicals.

274.—Ex. 1. Multiply $5a\sqrt{8}$ by $2\sqrt{6b}$.

SOLUTION. Multiplying the coefficients $5a$ and 2 together, we obtain $10a$; multiplying the radical parts $\sqrt{8}$ and $\sqrt{6b}$ together, we obtain $\sqrt{48b}$; and we have as the entire product $10a\sqrt{48b}$, which, reduced to its simplest form, is $40a\sqrt{3b}$.

$$5a\sqrt{8}\times 2\sqrt{6b}=$$
$$5a\times 2\times\sqrt{8}\times\sqrt{6b}=$$
$$10a\sqrt{48b}=10a\sqrt{16\times 3b}$$
$$=40a\sqrt{3b}$$

2. Multiply $\sqrt{8}$ by $\sqrt[3]{16}$.

SOLUTION. Reducing to radicals of the same degree, we obtain $\sqrt[6]{8^3}$ and $\sqrt[6]{16^2}$, and, multiplying and reducing, we obtain $4\sqrt[6]{32}$.

$$\sqrt{8}\times\sqrt[3]{16}=8^{\frac{3}{6}}\times 16^{\frac{2}{6}}$$
$$\sqrt[6]{8^3}\times\sqrt[6]{16^2}=\sqrt[6]{512\times 256}$$
$$\sqrt[6]{4^6\times 32}=4\sqrt[6]{32}$$

3. Multiply $5\sqrt{a}$ by $3\sqrt[3]{a}$. *Ans.* $15\sqrt[6]{a^5}$.

275. Rule for Multiplication of Radicals.—*Reduce the radical parts, if necessary, to those of the same degree; then multiply the coefficients together for the coefficient of the product, and the parts under the radical signs for the radical part.*

If polynomials contain radicals, the process is the same as in multiplication of polynomials.

PROBLEMS.

1. Multiply $3\sqrt{15}$ by $\sqrt{6}$. *Ans.* $9\sqrt{10}$.

2. Multiply $\sqrt{9x}$ by $\sqrt{x}$.

3. Multiply $5\sqrt[3]{6}$ by $3\sqrt[3]{4}$. *Ans.* $30\sqrt[3]{3}$.

4. Multiply $6\sqrt[3]{a}$ by $\sqrt[3]{2c}$.

5. Multiply $\frac{2}{3}\sqrt[3]{7a}$ by $\frac{4}{7}\sqrt[3]{4a^2}$. *Ans.* $\frac{8a}{21}\sqrt[3]{28}$.

6. Multiply $\frac{1}{4}\sqrt{6}$ by $\frac{2}{15}\sqrt{9}$.

7. Multiply $\sqrt{ax}$ by $3\sqrt[4]{ax}$. *Ans.* $3\sqrt[4]{a^3x^3}$.

8. Multiply $\sqrt{\frac{2xy}{3a}}$ by $\sqrt{\frac{3bx}{2y}}$.

9. Multiply $\sqrt[m]{a}$ by $\sqrt[n]{a}$. *Ans.* $\sqrt[mn]{a^{m+n}}$.

10. Multiply $a^{\frac{1}{2}}+x^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-x^{\frac{1}{2}}$.

CASE VII.

Division of Radicals.

276.—Ex. 1. Divide $8\sqrt{108}$ by $2\sqrt{6}$.

Solution. Dividing the coefficient of the dividend by the coefficient of the divisor, we obtain 4; dividing the radical part of the dividend by the radical part of the divisor, we obtain $\sqrt{18}$; and have as the entire quotient $4\sqrt{18}$, which, reduced to its simplest form, is $12\sqrt{2}$.

$$\frac{8\sqrt{108}}{2\sqrt{6}}=4\sqrt{18}$$
$$=4\sqrt{9\times 2}=12\sqrt{2}$$

2. Divide $\sqrt{12}$ by $\sqrt[3]{24}$.

Solution. Reducing the given radicals to radicals of the same degree, we obtain $\sqrt[6]{12^3}$ and $\sqrt[6]{24^2}$, and dividing, we obtain $\sqrt[6]{3}$.

$$\frac{\sqrt{12}}{\sqrt[3]{24}}=\frac{\sqrt[6]{12^3}}{\sqrt[6]{24^2}}=\sqrt[6]{3}$$

3. Divide $4a^2\sqrt{cd}$ by $2a\sqrt{c}$. *Ans.* $2a\sqrt{d}$.

277. **Rule for Division of Radicals.**—*Reduce the radical parts, if necessary, to those of the same degree, divide the coefficient of the dividend by the coefficient of the divisor for the coefficient of the quotient, and divide the radical of the dividend by the radical of the divisor for the radical part of the quotient.*

PROBLEMS.

1. Divide $12\sqrt{a}$ by $3\sqrt[3]{a}$. *Ans.* $4\sqrt[6]{a}$.
2. Divide $\sqrt{54}$ by $5\sqrt{2}$.
3. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$. *Ans.* $2\sqrt[3]{4}$.
4. Divide $2\sqrt[4]{19ax^2}$ by $\sqrt[4]{8a}$.
5. Divide $4\sqrt[3]{ax}$ by $3\sqrt{xy}$. *Ans.* $\frac{4}{3}\sqrt[6]{\frac{a^2}{xy^3}}$.
6. Divide $6\sqrt[3]{7a^7}$ by $3\sqrt[3]{2ac^2}$.
7. Divide $6\sqrt{a^4b}$ by $3ab^{-\frac{1}{2}}$. *Ans.* $2ab$.
8. Divide $\sqrt[5]{a}$ by $\sqrt[5]{b}$.
9. Divide $(a^2-x^2)^{\frac{1}{2}}$ by $\sqrt{a+x}$. *Ans.* $\sqrt{a-x}$.
10. Divide $\frac{1}{2}\sqrt[3]{\frac{1}{2}}$ by $\frac{1}{3}\sqrt[3]{\frac{1}{3}}$.

CASE VIII.

Involution of Radicals.

278.—Ex. 1. Involve $2a\sqrt{3b^3}$ to the second power.

SOLUTION. By the definition of involution, we take the given radical twice as a factor, and obtain $4a^2\sqrt{9b^6}$, which, reduced, is $12a^2b^3$.

$$(2a\sqrt{3b^3})^2 = 2a\sqrt{3b^3}\times 2a\sqrt{3b^3}$$

$$= 4a^2\sqrt{9b^6} = 4a^2\times 3b^3 = 12a^2b^3$$

14

2. Involve $5a\sqrt[3]{x}$ to the third power. *Ans.* $125a^3x$.

279. Rule for Involution of Radicals.—*Involve the rational and radical parts to the required power, and reduce the result to its simplest form.*

When the quantity to be involved is affected by a fractional exponent, the involution may be performed by multiplying the exponent of each letter by the exponent of the required power.

Dividing the index of the root produces the same effect as multiplying the fractional exponent.

PROBLEMS.

1. What is the second power of $a\sqrt{3}$? *Ans.* $3a^2$.

2. What is the third power of $5\sqrt[3]{a^2x}$?

3. Involve $\frac{2}{3}\sqrt{3}$ to the third power. *Ans.* $\frac{8}{9}\sqrt{3}$.

4. Involve $3\sqrt[3]{2a^2b}$ to the fourth power.
Ans. $162a^2b\sqrt[3]{2a^2b}$.

5. Involve $\frac{1}{6}\sqrt[3]{6a^3}$ to the fourth power.

6. What is the fourth power of $\frac{1}{2}\sqrt{4(a^2-x^2)}$?
Ans. $a^4-2a^2x^2+x^4$.

7. What is the sixth power of $(a+b)^{\frac{1}{3}}$?

CASE IX.

Evolution of Radicals.

280.—Ex. 1. Find the square root of $4a^2\sqrt{9b^6}$.

$$\sqrt{4a^2\sqrt{9b^6}} = \pm 2a\sqrt{3b^3}$$

SOLUTION. Since the root of a quantity is equal to the product of the roots of its factors (Art. 260), we take the square root of the coefficient, which is $\pm 2a$, and the square root of the quantity under the sign, which is $\sqrt{3b^3}$; uniting these, we have, for the entire root, $\pm 2a\sqrt{3b^3}$.

2. Find the cube root of $54\sqrt{3a}$.

SOLUTION. The coefficient, which is not a perfect cube, is composed of the factors 27 and 2, the former of which, 27, is a perfect cube. We take the cube root of 27, which is 3. We square the 2, since it is not a perfect cube, and introduce it as factor under the sign. The $12a$ under the sign is not a perfect square; hence, we denote its root by multiplying the index of the sign by the index of the required root. We have then for the entire root $3\sqrt[6]{12a}$.

$$\sqrt[3]{54\sqrt{3a}}=\sqrt[3]{27\times 2\sqrt{3a}}$$
$$=\sqrt[3]{27}\times\sqrt[3]{2\sqrt{3a}}=3\sqrt[3]{2\sqrt{3a}}$$
$$=3\sqrt[3]{\sqrt{12a}}=3\sqrt[6]{12a}$$

281. Rule for the Evolution of Radicals.—*Extract the required root of the rational and radical parts, if possible, and reduce the result to its simplest form.*

If the rational part is not a perfect power, introduce it under the radical sign; and if the radical part is not a perfect power, multiply its index by the index of the required root.

When the quantity to be evolved is affected by a fractional exponent, the evolution may be performed by dividing the exponent of each letter by the index of the required root.

PROBLEMS.

1. Extract the square root of $9\sqrt[3]{3}$. *Ans.* $\pm 3\sqrt[6]{3}$.

2. Extract the square root of $25a^4b^2c$.

3. Extract the cube root of $\frac{1}{27}\sqrt{3a}$. *Ans.* $\frac{1}{3}\sqrt[6]{3a}$.

4. What is the cube root of $\frac{8}{27}a^4$?

5. What is the square root of $a^2\sqrt{\frac{a}{x}}$? *Ans.* $\pm\frac{a}{x}\sqrt[4]{ax^3}$.

6. What is the cube root of $\frac{x}{3}\sqrt{\frac{x}{3}}$?

7. What is the cube root of $27a^{\frac{2}{3}}b^{\frac{3}{2}}$? *Ans.* $3a^{\frac{2}{9}}b^{\frac{1}{2}}$.

8. What is the fourth root of $a^4b^{-5}c^{\frac{2}{3}}$?

SECTION XXXIII.

RATIONALIZATION.

282. Rationalization is the process of removing the radical sign from a quantity.

This transformation is often of great utility, especially in finding the numerical values of fractional radicals.

CASE I.

Rationalization of any Monomial Surd.

283.—Ex. 1. Rationalize $\sqrt{a}$ and $a^{\frac{1}{3}}$.

$$\sqrt{a} \times \sqrt{a} = a$$
$$a^{\frac{1}{3}} \times a^{\frac{2}{3}} = a$$

SOLUTION. Multiplying $\sqrt{a}$ by $\sqrt{a}$, we have a, which is a rational quantity.

Multiplying $a^{\frac{1}{3}}$ by $a^{\frac{2}{3}}$ (which is the same quantity, with such a fractional exponent as will make the sum of the fractional exponents equal to 1), we have a, which is a rational quantity.

2. What factor will rationalize $a^{\frac{4}{7}}$? *Ans.* $a^{\frac{3}{7}}$.

284. Rule for the Rationalization of any Monomial Surd.—*Multiply the surd by the same quantity with such a fractional exponent as, when added to the given exponent, shall be equal to one.*

PROBLEMS.

1. What factor will rationalize $\sqrt[3]{a^2}$? *Ans.* $\sqrt[3]{a}$.

2. What factor will rationalize $5\sqrt{a^3}$?

3. What factor will rationalize $3\sqrt[3]{a^2b}$? *Ans.* $\sqrt[3]{ab^2}$.

4. What factor will rationalize $\sqrt[5]{(a-b)^2}$?

CASE II.

Rationalization of Binomial Surds of the Second Degree.

285.—Ex. 1. Rationalize $\sqrt{a}+\sqrt{b}$.

SOLUTION. The product of the sum and the difference of two quantities is equal to the difference of their squares; hence, $\sqrt{a}+\sqrt{b}$ multiplied by $\sqrt{a}-\sqrt{b}$ is equal to $a-b$, which is rational.

$$\begin{array}{r} \sqrt{a}+\sqrt{b} \\ \underline{\sqrt{a}-\sqrt{b}} \\ a-b \end{array}$$

2. What factor will rationalize $\sqrt{a}-\sqrt{b}$? *Ans.* $\sqrt{a}+\sqrt{b}$.

286. Rule for Rationalization of Binomial Surds of the Second Degree.—*Multiply the binomial by the same expression with the sign of one of the terms changed.*

PROBLEMS.

1. What factor will rationalize $a+3\sqrt{8}$? *Ans.* $a-3\sqrt{8}$.
2. Rationalize $\sqrt{7}-\sqrt{5}$.
3. What factor will rationalize $\sqrt{5}-\sqrt{a}$? *Ans.* $\sqrt{5}+\sqrt{a}$.
4. Rationalize $5+\sqrt{3}$.

CASE III.

Rationalization of either Term of a Fraction.

287.—Ex. 1. Reduce $\frac{a}{\sqrt{c}}$ to a fraction whose denominator is rational.

SOLUTION. Multiplying both terms of the fraction by $\sqrt{c}$, which does not change the value of the fraction, we obtain $\frac{a\sqrt{c}}{c}$, in which the denominator is rational.

$$\frac{a}{\sqrt{c}}\times\frac{\sqrt{c}}{\sqrt{c}}=\frac{a\sqrt{c}}{c}$$

2. Rationalize the denominator of $\frac{2}{\sqrt{3}}$. *Ans.* $\frac{2\sqrt{3}}{3}$.

288. Rule for the Rationalization of either Term of a Fraction.—*Multiply both terms of the fraction by such a factor as will render either term rational, as may be required.*

PROBLEMS.

1. Rationalize the numerator of $\frac{\sqrt{a}}{\sqrt{c}}$. *Ans.* $\frac{a}{\sqrt{ac}}$.

2. Rationalize the denominator of $\frac{b}{\sqrt[3]{a}}$.

3. Rationalize the denominator of $\frac{\sqrt{3}}{\sqrt[3]{3}}$. *Ans.* $\frac{\sqrt[6]{3^7}}{3}=\sqrt[6]{3}$.

4. Rationalize the denominator of $\frac{a}{3+\sqrt{a}}$.

5. What factor will rationalize the denominator of $\frac{b}{\sqrt[n]{a}}$?

6. What factor will rationalize the denominator of $\frac{\sqrt{2}}{3-\sqrt{2}}$?

SECTION XXXIV.

RADICAL EQUATIONS SOLVED LIKE SIMPLE EQUATIONS.

289. Radical Equations are those which contain the unknown quantity under the radical sign.

In the process of solving such equations, it will be found necessary first to rationalize the surd expression of the unknown quantity, and then to find its value.

290.—Ex. 1. Find the value of x in $\sqrt{x}+4=9$.

SOLUTION. Transposing and uniting (1), we obtain (2). Squaring both members of (2), we obtain (3), or $x=25$.

$$\sqrt{x}+4=9 \quad (1)$$
$$\sqrt{x}=5 \quad (2)$$
$$x=25 \quad (3)$$

2. Find the value of x in $\sqrt{x+9}=\sqrt{x}+1$.

Given, $\sqrt{x+9}=\sqrt{x}+1$

Squaring, $x+9=x+2\sqrt{x}+1.$

Transposing, $-2\sqrt{x}=-8$

Dividing by -2, $\sqrt{x}=4$

Squaring, $x=16$

291. Rule for Solving Radical Equations.—*Transpose the terms, so that the radical part shall stand as one side of the equation; then involve each side to a power of the same degree as the radical. If there be still a radical part, transpose and involve as before; and, finally, find the value of the unknown quantity as in ordinary simple equations.*

PROBLEMS.

1. Given $\sqrt{x+4}=8$, to find x. *Ans.* $x=60.$

2. Given $\sqrt{3x+4}=5$, to find x.

3. Given $\sqrt{\frac{2x}{3}}+5=7$, to find x. *Ans.* $x=6.$

4. Given $5+\sqrt{7x+8}=13$, to find x.

5. Given $\sqrt{12+x}=2+\sqrt{x}$, to find x. *Ans.* $x=4.$

6. Given $\sqrt{x-32}=16-\sqrt{x}$, to find x.

7. Given $\sqrt{ax-bx}=c$, to find x. *Ans.* $x=\frac{c^2}{a-b}.$

8. Given $\sqrt{x-16}=8-\sqrt{x}$, to find x.

9. Given $2+\sqrt{3x}=\sqrt{4+5x}$, to find x. *Ans.* $x=12.$

10. Given $\sqrt{x+11}-5=\sqrt{x-4}$, to find x.

11. Given $\frac{x+ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$, to find x. *Ans.* $x=\frac{1}{1+a}$.

12. Given $7\sqrt{\frac{2}{3}x-6}=14$, to find x.

13. Given $3+5\sqrt{x+4}=28$, to find x. *Ans.* $x=21$.

14. Given $(10x+3)^{\frac{1}{2}}-2=5$, to find x.

SECTION XXXV.

REVIEW PROBLEMS.

292.—Ex. 1. Find the nth power of a^3b^2. *Ans.* $a^{3n}b^{2n}$.

2. Find the square of $-\frac{3ax^2}{2b}$.

3. Expand $\left(a-\frac{1}{a}-1\right)^2$. *Ans.* $a^2-2a+\frac{1}{a^2}+\frac{2}{a}-1$.

4. Find the value of $\sqrt[3]{125a^6b^3}$.

5. Reduce $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$ and $b^{\frac{2}{3}}$ to the same degree.
Ans. $\sqrt[6]{a^3}$; $\sqrt[6]{a^2}$; $\sqrt[6]{b^4}$.

6. Place the coefficient of $3xy\sqrt[3]{2x^2y}$ under the sign.

7. Add $\sqrt{16a}$ and $\sqrt{4a}$. *Ans.* $6\sqrt{a}$.

8. From $a\sqrt[3]{2a}$ take $bc\sqrt[3]{2a}$.

9. Multiply $2\sqrt{b}$ by $3\sqrt[3]{b^2}$. *Ans.* $6\sqrt[6]{b^7}$.

10. From $-a^{-\frac{1}{n}}$ take $-2a^{-\frac{1}{n}}$.

11. Divide $24x\sqrt{ac}$ by $6\sqrt{a}$. *Ans.* $4x\sqrt{c}$.

12. What is the nth power of $a^{\frac{1}{n}}$?

13. Make the numerator of $\frac{\sqrt{a}}{\sqrt{x}}$ rational. *Ans.* $\frac{a}{\sqrt{ax}}$.

14. What is the value of x in $4\sqrt{\frac{x}{5}}=8$?

15. Find the square root of $24\sqrt{3c}$. *Ans.* $2\sqrt[4]{108c}$.

16. Make the denominator of $\frac{\sqrt{2}}{3-\sqrt{2}}$ rational.

17. Reduce $2\sqrt{128a^3x^5}$ to its simplest form.

18. What is the value of x in $\sqrt{x}-a\sqrt{x}=\frac{1}{\sqrt{x}}$?

19. What is the value of $\frac{1}{\sqrt{2}}$? *Ans.* $.707+$

20. What is the value of $\frac{\sqrt{3}}{\sqrt{7}}$ to two places of decimals?

Test Questions.

293.—1. What is a *Perfect Power?* An imperfect power? What is involution? What are the given principles of involution?

2. What is the *Rule* for the involution of monomials? For the involution of polynomials? By what principles may any polynomials be squared?

3. What is *Evolution?* In fractional exponents what does the numerator denote? What does the denominator denote? What are the given principles of evolution?

4. What is the *Rule* for the evolution of monomials? For extracting the square root of polynomials? How can you find the square root of a trinomial which is a perfect square?

5. What are the given *principles* of the extraction of the square root of numbers? What is the rule for the extraction of the square root of numbers?

6. What is a *Radical?* The coefficient of a radical? The degree? What are similar radicals? What is a rational quantity? An irrational quantity? What are the given principles of radicals?

7. When is a radical in its *simplest form?* What is the rule for reducing a radical to its simplest form? For reducing a rational quantity to the form of a radical?

8. What is the *Rule* for reducing radicals to the same degree? For the addition of radicals? For the subtraction of radicals? For the multiplication of radicals? For the division of radicals? For the involution of radicals? For the evolution of radicals?

9. What is *Rationalization?* What is the rule for the rationalization of any monomial surd? For the rationalization of binomial surds? Of either term of a fraction?

10. What are *Radical Equations?* What is the rule for solving radical equations?

SECTION XXXVI.

PURE QUADRATIC EQUATIONS.

294.—Ex. 1. The equation $3x+2=23$ contains what power of the unknown quantity?

2. In the equation $x^2+2x=99$, what is the highest power of the unknown quantity?

3. The equation $5x-6=54$ contains how many powers of the unknown quantity?

4. If the equation $x=3$ be multiplied by x, what will the equation become?

5. If the equation $x=3$ be multiplied by $2x$, what will the equation become?

6. If the equation $x^2=3x$ be divided by x, what equation will express the result?

7. If the equation $2x^2=6x$ be divided by x, what equation will express the result?

8. If $x^2=81$, what will be the value of x? If $x^2=3x$, what will be the value of x?

9. If the number of my books were multiplied by the number, the product would be 8 times the number of my books. What is the number?

10. The square of a number is 9 times the number; what is the number?

11. What is the value of x in the equation $3x^2=12x$? In the equation $5x^2=35x$?

Definitions.

295. A **Quadratic Equation** is an equation of the second degree, or one in which the second power is the highest power of the unknown quantity.

Thus, $x^2=64$ and $x^2-2x=32$ are quadratic equations.

296. A **Pure Equation** is an equation which contains but one power of the unknown quantity.

Thus, $x^2=81$ and $x+2=21$ are pure equations.

297. A **Pure Quadratic Equation** is an equation which contains the second power, and no other power, of the unknown quantity.

Thus, $2x^2+8=24$ is a pure quadratic equation.

A pure quadratic equation is called an **incomplete** equation of the second degree.

298. Principles.—1. *Every pure quadratic equation can be reduced to the general form $ax^2=b$.*

Thus, the equation $2x^2+8=40$ may become $2x^2=32$. Now, if we let a denote 2, the coefficient of x^2, and b denote 32, the value of $2x^2$, we obtain $ax^2=b$.

2. *Every pure quadratic equation has two roots of equal numerical values, but with different signs.*

For, an even root of a positive quantity is either positive or negative (Art. 244–2).

299.—Ex. 1. What is the value of x in the equation $x^2+7=88$?

SOLUTION. Transposing and uniting, we obtain (2), and extracting the square root of both members of the equation, we obtain (3), or $x=\pm 9$.

$$x^2+7=88 \quad (1)$$
$$x^2=81 \quad (2)$$
$$x=\pm 9 \quad (3)$$

2. What is the value of x in the equation $\frac{x^2}{16}-\frac{x^2-3}{5}=\frac{1}{20}$?

SOLUTION. Clearing of fractions, we obtain (2); transposing and uniting, we obtain (3); dividing by -11, we obtain (4), and extracting the square root, we obtain (5), or $x=\pm 2$.

$$\frac{x^2}{16}-\frac{x^2-3}{5}=\frac{1}{20} \quad (1)$$
$$5x^2-16x^2+48=4 \quad (2)$$
$$-11x^2=-44 \quad (3)$$
$$x^2=4 \quad (4)$$
$$x=\pm 2 \quad (5)$$

300. Rule for Solving a Pure Quadratic Equation.—*Find the value of the square of the unknown quantity, and then extract the square root of each member of the equation.*

PROBLEMS.

1. Given $x^2-15=10$, to find x. *Ans.* $x=\pm 5$.

2. Given $x^2+9=58$, to find x.

3. Given $\frac{3x^2}{4}-2=10$, to find x. *Ans.* $x=\pm 4$.

4. Given $x^2-90=31$, to find x.

5. Given $\frac{x}{2}-\frac{4}{x}=\frac{x}{4}$, to find x. *Ans.* $x=\pm 4$.

6. Given $x^2+ab=5x^2$, to find x.

7. Given $\frac{1}{2x^2}=\frac{9}{4x^2}-7$, to find x.

8. Given $\frac{5}{4+x}+\frac{5}{4-x}=\frac{8}{3}$, to find x. *Ans.* $x=\pm 1$.

9. Given $\frac{1}{8}(3x^2+5)-\frac{1}{3}(x^2+21)=39-5x^2$, to find x.

SECTION XXXVII.

PROBLEMS PRODUCING PURE QUADRATIC EQUATIONS.

301.—Ex. 1. If a certain number increased by 6 is multiplied by the same number diminished by 6, the product is 64. What is the number?

SOLUTION.

Let $x=$ *the number;*

then, $x+6=$ *one factor;*

and $x-6=$ *the other.*

By conditions, $(x+6)(x-6)=64$

or, $x^2-36=64$

Transposing and uniting, $x^2=100$

Extracting the square root, $x=\pm 10$

Using the positive value, we have the answer 10.

2. What number is that the fourth part of whose square being subtracted from 8, leaves a remainder equal to 4?

3. The distance to a certain village is such that if 96 be subtracted from the square of the number of miles, the remainder will be 48. What is the distance? *Ans. 12 miles.*

4. Two men were talking of their ages; one said that he was 94 years old. "Then," said the younger, "if the number of years in your age, plus the number in mine, be multiplied by the difference of those numbers, the product will be 8512." What is the age of the younger?

5. An army was drawn up with 5 more men in depth than in front; but when the front was increased by 845 men, the army was arranged in 5 lines. What was the number of men in the army? *Ans. 4550.*

6. What two numbers are such that $\frac{2}{9}$ of the greater is equal to their difference, and the difference of their squares is 128?

7. There is a rectangular field whose length is $\frac{6}{5}$ of its breadth. A part of this field equal to $\frac{1}{6}$ of the whole having been measured off, there remain 625 square rods. What are the dimensions of the field? *Ans. Length, 30 rods; width, 25 rods.*

8. A man being asked how much money he had, replied, that if the square of the number of dollars he had were multiplied by 7, the product would be 1575. How many dollars had he?

9. William bought two pieces of cloth, which together measured 36 yards. Each piece cost as many shillings a yard as there were yards in it, and the entire cost of one piece was 4 times that of the other. How many yards were there in each piece? *Ans. 24 in one; 12 in the other.*

10. There are in a certain house two square parlors. The side of the larger is $1\frac{1}{2}$ times that of the smaller, and it takes 180 square feet more of carpeting to cover the floor of the one than of the other. What is the length of one side of each room? *Ans. 18 and 12 feet, respectively.*

SECTION XXXVIII.

AFFECTED QUADRATIC EQUATIONS.

302.—Ex. 1. What powers of the unknown quantity are contained in the equation $x^2-8x+16=33$?

2. In the equation $x^2+8x=20$, what is the highest power of the unknown quantity? Of what degree, then, is that equation?

3. If $x^2+10x+25$ is the square of $x+5$, what is the square root of $x^2+10x+25$?

4. In x^2+10x what term is wanting to make it the square of $x+5$?

5. In x^2+6x what term is wanting to make it the square of $x+3$?

6. In x^2+10x what term is wanting to make it a perfect square?

7. In x^2+6x what term is wanting to make it a perfect square?

Definitions.

303. An **Affected Quadratic Equation** is an equation which contains both the square and the first power of the unknown quantity.

Thus, $x^2+\frac{18x}{3}=27$ is an affected quadratic equation.

An affected quadratic equation is sometimes called a **Complete Equation** of the second degree.

304. Principles.—1. *Every affected quadratic equation can be reduced to the general form $ax^2+bx=c$.*

Thus, the equation $x^2+\frac{18x}{3}=27$, when cleared of fractions, becomes $3x^2+18x=81$. Denoting 3, the coefficient of x^2, by a; 18, the coefficient of x, by b; and 81, the value of $3x^2+18x$, by c, we obtain $ax^2+bx=c$. Now, as a may stand for any coefficient of x^2, b for any coefficient of x, and c for any value of the terms containing the unknown quantity, the equation $ax^2+bx=c$ is a general form for affected quadratic equations.

2. *Affected quadratic equations depend for their solution on the form of a squared binomial.*

Thus, $(x+a)^2=x^2+2ax+a^2$. Here a^2 is evidently the square of half of $2a$, the coefficient of x. If, then, we have the expression x^2+2ax, which is not a perfect square, we can make it a perfect square by adding to it the square of half the coefficient of x. Hence, if we have the equation $x^2+2ax=b$, we add a^2 to each member, thus making the first member a perfect square, and preserving the equality of the two members, and have the equation $x^2+2ax+a^2=b+a^2$.

This process of adding the square of half the coefficient of x to both members of the equation is called *completing the square.*

305.—Ex. 1. What is the value of x in the equation $x^2+8x=9$?

SOLUTION. If the square of half the coefficient of x, or 4^2, be added to the first member, it will be in the form of a squared binomial. Adding, then, 4^2, or 16, to the first member to complete the square, and 16 to the second member to preserve the equality, we obtain (2). Extracting the square root of both members, we obtain (3). Transposing, we obtain (4); whence (5), or $x=+1$, and (6), or $x=-9$.

$$x^2+8x=9 \quad (1)$$
$$x^2+8x+4^2=25 \quad (2)$$
$$x+4=\pm 5 \quad (3)$$
$$x=\pm 5-4 \quad (4)$$
$$x=+1 \quad (5)$$
$$x=-9 \quad (6)$$

Two values, therefore, of the unknown quantity are obtained, and it will be found, on substitution, that the value of x^2+8x is 9, whether 1 or -9 be put in the place of x.

2. What is the value of x in the equation $x^2-8x=-12$?

SOLUTION. Completing the square by adding 4^2, the square of half the coefficient of x, to each member, we obtain (2). Extracting the square root of both members, we obtain (3). Transposing, we obtain (4); whence (5), or $x=6$, and (6), or $x=2$.

$$x^2-8x=-12 \quad (1)$$
$$x^2-8x+4^2=4 \quad (2)$$
$$x-4=\pm 2 \quad (3)$$
$$x=\pm 2+4 \quad (4)$$
$$x=6 \quad (5)$$
$$x=2 \quad (6)$$

3. What is the value of x in the equation $-4x^2+7x=-2$?

SOLUTION. Dividing both members by -4, we obtain (2). Completing the square, we obtain (3). Extracting the square root, we obtain (4). Transposing, we obtain (5), whence (6), or $x=2$, and (7), or $x=-\frac{1}{4}$.

$$-4x^2+7x=-2 \qquad (1)$$
$$x^2-\frac{7x}{4}=\frac{2}{4} \qquad (2)$$
$$x^2-\frac{7}{4}x+\left(\frac{7}{8}\right)^2=\frac{81}{64} \qquad (3)$$
$$x-\frac{7}{8}=\pm\frac{9}{8} \qquad (4)$$
$$x=\pm\frac{9}{8}+\frac{7}{8} \qquad (5)$$
$$x=2 \qquad (6)$$
$$x=-\frac{1}{4} \qquad (7)$$

4. What is the value of x in the equation $x^2+6x=-8$?

SOLUTION. Completing the square, we obtain (2). Extracting the square root, we obtain (3). Transposing, we obtain (4). Whence (5), or $x=-2$, and (6), or $x=-4$.

$$x^2+6x=-8 \qquad (1)$$
$$x^2+6x+3^2=1 \qquad (2)$$
$$x+3=\pm 1 \qquad (3)$$
$$x=\pm 1-3 \qquad (4)$$
$$x=-2 \qquad (5)$$
$$x=-4 \qquad (6)$$

From these solutions it appears that the two roots of an affected quadratic equation may have different signs or may both have the same sign.

5. What is the value of x in the equation $5x^2-20x=-15$?

Ans. $x=3$, *or* 1.

6. What is the value of x in the equation $11x^2-88x=99$?

Ans. $x=9$, *or* -1.

306. Rule for Solving Affected Quadratic Equations.—*Reduce the given equation, if necessary, to the general form,* $ax^2+bx=c$.

Divide each member of the equation by the coefficient of x^2, *and complete the square by adding to each member the square of one-half the coefficient of* x *in the resulting equation.*

Extract the square root of both members, and solve the resulting equation.

In the general form, bx and c may be either positive or negative, integral or fractional (Art. 304).

The square root of a negative quantity is impossible; therefore, when the sign of the term containing x^2 is negative, change the signs of all the terms in the equation, which is equivalent to dividing by -1.

PROBLEMS.

1. Given $x^2+6x=40$, to find x. *Ans.* $x=4$, *or* -10.

2. Given $3x^2-12x=96$, to find x. *Ans.* $x=8$, *or* -4.

3. Given $x^2-3x=-2$, to find x. *Ans.* $x=1$, *or* 2.

4. Given $x^2+6=5x$, to find x.

5. Given $7x^2+28x-224=0$, to find x. *Ans.* $x=-8$, *or* 4.

6. Given $3x-54=-x^2$, to find x.

7. Given $x=2+\frac{5}{4x}$, to find x. *Ans.* $x=2\frac{1}{2}$, *or* $-\frac{1}{2}$.

8. Given $\frac{9}{x}-\frac{x}{3}=2$, to find x.

9. Given $4x+22=\frac{12-x}{x-3}$, to find x. *Ans.* $x=-6$, *or* $3\frac{1}{4}$.

10. Given $x+\frac{1}{x-3}=5$, to find x. *Ans.* $x=4$, *or* 4.

Here, the two roots of the equation are alike, both in signs and numerical values. Hence, the equation is said to have *equal roots.*

11. Given $x^2+10x=24$, to find x. *Ans.* $x=2$, *or* -12.

12. Given $\frac{2x+11}{x}=5-\frac{x-5}{3}$, to find x.

13. Given $x(x+3)=180$, to find x. *Ans.* $x=12$, *or* -15.

14. Given $x^2+3x-54=0$, to find x.

15. Given $2x^2+12x=-16$, to find x. *Ans.* $x=-2$, *or* -4.

307. Another Method of completing the square, which has the advantage of avoiding fractions, is often to be preferred to the preceding method.

Let the equation be brought to the form

$$ax^2+bx=c,$$

in which a and b are whole numbers and prime to each other, and c is either a whole number or a fraction.

Multiply each member of this equation by a, the coefficient of x^2, and by 4, the smallest even square number, and it becomes

$$4a^2x^2+4abx=4ac,$$

in which the first term is a perfect square, and the second term is divisible by 2.

Regard $4a^2x^2+4abx$ as the first two terms of the square of a binomial. The first term of the binomial must be the square root of $4a^2x^2$, or $2ax$. The $4abx$ must be twice the product of the first term of the binomial by the second; hence, the quotient of $4abx$ divided by $4ax$, which is b, must be the other term of the binomial.

Adding the square of b, or b^2, to the first member of the above equation to make it a perfect square, and to the other member to preserve the equality, we have the equation,

$$4a^2x^2+4abx+b^2=4ac+b^2.$$

The first member of this equation is a complete square, and its several terms are whole numbers.

Extracting the square root, we have

$$2ax+b=\pm\sqrt{4ac+b^2},$$

whence, by transposing b and dividing by $2a$, we have

$$x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}.$$

When b is an even number, $\frac{b}{2}$ will be a whole number, and it will be sufficient to multiply each member by a, and then add $\left(\frac{b}{2}\right)^2$ to each member.

Hence, the following rule:—

308. Rule for Solving Affected Quadratics.—*Reduce the given equation to the form* $ax^2+bx=c$, *in which the coefficients* a *and* b *are integral and prime to each other.*

If the coefficient of x *be odd, multiply the equation by four times the coefficient of* x^2, *and add the square of the coefficient of* x *to both members; or,*

If the coefficient of x *be even, multiply the equation by the coefficient of* x^2, *and add the square of one-half of the coefficient of* x *to both members.*

Extract the square root of both members, and solve the resulting equation.

PROBLEMS.

1. Given $x^2+6x+4=22-x$, to find x.

SOLUTION.

Given,	$x^2+6x+4=22-x$
By transposition,	$x^2+7x=18$
Multiplying by 4,	$4x^2+28x=72$
Completing the square,	$4x^2+28x+7^2=121$
Extracting the square root,	$2x+7=\pm 11$
whence,	$x=\dfrac{\pm 11-7}{2}$
or,	$x=2$
and,	$x=-9$

2. Given $x^2-5x=-6$, to find x. *Ans.* $x=3$, *or* 2.

3. Given $x^2-13x=68$, to find x.

4. Given $x^2+25x=-100$, to find x. *Ans.* $x=-5$, *or* -20.

5. Given $x^2-\dfrac{3x}{2}=27$, to find x.

6. Given $2x^2-7x+3=0$, to find x. *Ans.* $x=3$, *or* $\dfrac{1}{2}$.

7. Given $3x^2+10x=57$, to find x.

8. Given $2x^2-12x+3=83$, to find x.

9. Given $4x^2-12x+9=0$, to find x.

10. Given $x^2-6x=2$, to find x. *Ans.* $x=3\pm\sqrt{11}$.

11. Given $\frac{x+1}{x-1}-\frac{x-1}{x+1}=1$, to find x.

12. Given $\frac{x^2}{2}-4x=7$, to find the approximate values of x to three orders of decimals. *Ans.* $x=9.477$, *or* -1.477.

13. Given $2\sqrt{x^2-4x}+4x=1$, to find x. *Ans.* $x=-\frac{1}{2}$, *or* $-\frac{1}{6}$.

14. Given $x+5=\sqrt{x+5}+6$, to find x.

15. Given $2x^2+x=11$, to find the approximate values of x to three orders of decimals. *Ans.* $x=2.108$, *or* -2.608.

16. Given $7x^2-63x+56=0$, to find x.

17. Given $z^2-8z+20=0$, to find z. *Ans.* $z=4\pm\sqrt{-4}$.

18. Given $y^2-2my=b^2$, to find y. *Ans.* $y=m\pm\sqrt{b^2+m^2}$.

309. Any equation in the **Quadratic Form**—that is, any equation which contains but two powers of the unknown quantity, the exponent of the one power being twice that of the other—can be solved by the application of the rules given for the solution of affected quadratics.

For, all such equations take, or can be made to take, the form

$$ax^{2n}+bx^n=c,$$

in which ^{n} may have any value whatever, and the two terms of the first member are so related that we can supply the third term, in order to complete the square of a binomial.

Ex. 1. Given $x^6-7x^3=8$, to find x.

SOLUTION.

Given, $x^6-7x^3=8$

Completing the square, $x^6-7x^3+\left(\frac{7}{2}\right)^2=\frac{81}{4}$

Extracting the square root, $x^3-\frac{7}{2}=\pm\frac{9}{2}$

whence, $x^3=\frac{7}{2}\pm\frac{9}{2}$

or, $x^3=8$, *or* -1

Extracting the cube root, $x=2$ *or,* -1

2. Given $x^6-10x^3=459$, to find x. *Ans.* $x=3$, *or* $\sqrt[3]{-17}$.

3. Given $x^4-9x^2+20=0$, to find x.

SOLUTION.

Given, $x^4-9x^2+20=0$

Transposing, $x^4-9x^2=-20$

Completing the square, $x^4-9x^2+\left(\frac{9}{2}\right)^2=\frac{1}{4}$

Extracting the square root, $x^2-\frac{9}{2}=\pm\frac{1}{2}$

whence, $x^2=\frac{9}{2}\pm\frac{1}{2}$

or, $x^2=5$, *or* 4

Extracting the square root, $x=\pm\sqrt{5}$, *or* ± 4

4. Given $16x^4-12x^2+2=0$, to find x. *Ans.* $x=\pm\frac{1}{2}$, *or* $\pm\sqrt{\frac{1}{2}}$.

5. Given $4x^6-3240=12x^3$, to find x.

6. Given $2x^4-16x^2=18$, to find x. *Ans.* $x=\pm 3$, *or* $\pm\sqrt{-1}$.

SECTION XXXIX.

PROBLEMS PRODUCING AFFECTED QUADRATIC EQUATIONS.

310.—Ex. 1. What two numbers are such that their difference is 12, and their product 64?

SOLUTION.

$x =$ *one number;*

$x + 12 =$ *the other number.*

By conditions, $x(x+12) = 64$

or, $x^2 + 12x = 64$

Completing the square, $x^2 + 12x + 6^2 = 64 + 36 = 100$

Extracting the square root, $x + 6 = \pm 10$

whence, $x = -6 \pm 10 = 4$, *or* -16

and, $x + 12 = 16$, *or* -4

Hence, the two numbers are 4 and 16, or -16 and -4, as either pair of values satisfies the problem.

2. What two numbers are such that their sum is 15, and their product 54? *Ans. 9 and 6.*

3. What is that number from the square of which if we take 7 times the number, the remainder will be 44?

Ans. 11, or -4.

4. The ages of a man and his wife amount to 42 years, and the product of the numbers expressing their ages is 432. What is the age of each? *Ans. The man, 24 years; the wife, 18.*

5. A wholesale shoemaker received an unexpected order for 990 pairs of shoes, to be finished by the end of the month. He found that if these were divided equally among his men, each would have allotted to him 12 pairs more than he could get finished by the given time at his ordinary rate of working. He therefore engaged 54 more men, and got the work executed by the time specified. How many men had he at first?

6. A merchant sold a quantity of flour for \$39, and gained as many per cent. as equalled the number of dollars in the price of the flour. What was the price of the flour?

SOLUTION.

$x = \left\{ \begin{array}{l} \textit{the number of dollars in} \\ \textit{the price of the flour;} \end{array} \right.$

$x = $ *the rate of gain per cent.;*

$$x \times \frac{x}{100} = \frac{x^2}{100} = \textit{the gain.}$$

By conditions, $\frac{x^2}{100} = 39 - x$

whence, $x^2 + 100x = 3900$

Completing the square, $x^2 + 100x + (50)^2 = 3900 + 2500 = 6400$

Extracting the square root, $x + 50 = \pm 80$

whence, $x = 30$, *or* -130

The cost of the flour was \$30. The other value of x, not satisfying the conditions of the question, is not admissible.

7. A person invested a certain sum of money for goods, which he sold again for \$24, and thereby lost as many per cent. as equalled the number of dollars invested. How much did he invest? *Ans. \$40, or \$60.*

8. There is a rectangular field whose length exceeds its breadth by 20 rods, and its area is 6300 square rods. What are its length and breadth?

9. A man planted a rectangular field with 8400 trees at equal distances, having 50 trees more in the longer rows than in the shorter ones. What was the number of trees in each of the longer rows? *Ans. 120.*

10. A trader computes that, during the time he has been in business, he has made \$6300 clear profit. His neighbor, however, who has been three years less time in business, has made the same sum, owing to his clearing \$27 per annum more. How long is it since the first commenced business?

11. A farmer sold a number of tons of hay for \$112, and observed that if he had sold one ton more for the same money, each ton would have brought him \$2 less. Required the number of tons sold and the price per ton.

SOLUTION.

x = *the number of tons sold;*

$\frac{112}{x}$ = {*the number of dollars in the price per ton.*

By conditions, $\frac{112}{x+1}=\frac{112}{x}-2$

Clearing of fractions, $112x=112x+112-2x^2-2x$

whence, $x^2+x=56$

Completing the square, $4x^2+4x+1=224+1=225$

Extracting the square root, $2x+1=\pm 15$

whence, $x=7$, *or* -8

and, $\frac{112}{x}=16$, *or* -14

The number of tons sold was 7, and the price per ton was \$16. The negative value of x is not admissible.

12. The sum of \$144 was divided equally among a certain number of persons. If there had been two persons less, each would have received \$1 more. How many persons were there? *Ans. 18.*

13. A man bought a certain number of sheep for \$80. If he had bought 4 more sheep for the same money, they would each have cost him \$1 less. How many sheep did he buy?

14. Among a certain number of poor persons 110 bushels of coals were equally divided. If each person had received 1 bushel more, he would have received as many bushels as there were persons. Required the number of persons. *Ans. 11.*

15. A cistern is supplied with water by two pipes; by one of them it can be filled in 6 hours sooner than by the other, and by both together in 4 hours. Find the time in which each pipe will fill it.

SOLUTION.

x = *the number of hours in which one will fill it;*

$x+6$ = *the number of hours in which the other will fill it.*

By conditions, $\frac{1}{x}+\frac{1}{x+6}=\frac{1}{4}$

whence, $x=6$, *or* -4

and, $x+6=12$, *or* -2

The first will fill it in 6 hours, and the second in 12 hours. The negative value of x is not admissible.

16. A and B can perform a certain piece of work in $14\frac{2}{5}$ days, and A alone can perform it in 12 days less than B alone. Find the time in which A alone can perform it.

17. Two travellers, wishing to meet, set out from two towns, A and B, which are 120 miles distant from each other; the first goes 6 miles a day, and the other 1 mile a day more than the number of days in which they meet. In how many days will they meet? *Ans. 8.*

18. The length of a rectangle exceeds its breadth by 12, and the sum of the squares of the length and breadth is 20880. What are the sides of the rectangle and the area?

Ans. Breadth, 96; length, 108; area, 10368.

19. In a concert-room 800 persons are seated on benches of equal length. If there were 20 fewer benches, it would be necessary that two more persons should sit on each bench. Find the number of benches.

SOLUTION.

$x =$ *the number of benches;*

$\frac{800}{x} =$ { *the number of persons on one bench in first case;*

$\frac{800}{x-20} =$ { *the number of persons on one bench in second case.*

By conditions, $\frac{800}{x-20} - \frac{800}{x} = 2$

whence, $x = 100$, *or* -80

Rejecting the negative value, we have 100 as the number of benches.

20. Divide 10 into two parts, such that the product of the parts shall be 24.

21. Two detachments of infantry are ordered to a station which is 39 miles distant. They begin their march at the same time; but one party, by travelling one-fourth of a mile an hour more than the other, arrives one hour sooner. Required the rates of marching per hour. *Ans. 3* and *$3\frac{1}{4}$ miles.*

22. A had 40 yards of cloth, and B 90 yards, which they sold together for \$42. Now, A sold for \$1, a third of a yard more than B sold for the same money. How many yards did each sell for \$1?

23. The difference of two numbers is 2, and the difference of their cubes is 152. What are the numbers?

24. Two boys, John and William, start at the same time to walk a distance of 75 miles; but John walks $1\frac{1}{2}$ miles per hour faster than William, and finishes his journey $8\frac{1}{3}$ hours before him. How many miles per hour did each walk?

Ans. John, $4\frac{1}{2}$; *William, 3.*

SECTION XL.

SIMULTANEOUS QUADRATIC EQUATIONS.

311. A **Homogeneous Equation** is one in which the sum of the exponents of the unknown quantities in each term containing such quantities is the same.

Thus, $x^2-y^2=12$ and $x^2-xy+y^2=12$ are each homogeneous.

312. A **Symmetrical Equation** is one in which the unknown quantities are similarly involved.

Thus, $x^2+y^2=25$ and $x^2y-xy^2=6$ are each symmetrical.

313. Simultaneous Quadratic Equations (Art. 211) containing two unknown quantities cannot all be solved by the rules for quadratics.

The cases which can be treated in an elementary work are necessarily only such as can be solved by comparatively simple processes.

314.—Ex. 1. Given $4x+2y=18$ and $5xy=50$, to find x and y.

SOLUTION.

$$4x+2y=18 \quad (1)$$
$$5xy=50 \quad (2)$$

From (1), $x=\dfrac{9-y}{2}$ (3)

Substituting in (2), $\dfrac{45y-5y^2}{2}=50$ (4)

Clearing of fractions, $45y-5y^2=100$ (5)

or, $5y^2-45y=-100$ (6)

Dividing by 5, $y^2-9y=-20$ (7)

Completing the square, $4y^2-36y+81=1$ (8)

Extracting the square root, $2y-9=\pm 1$ (9)

whence, $y=\dfrac{9\pm 1}{2}$ (10)

Substituting in (2) *these values of* y, $y=5$, *or* 4 (11)

we have, $x=2$, *or* $2\frac{1}{2}$ (12)

Here one of the equations is simple and the other quadratic.

2. Given $2x-2y=11$ and $xy=20$, to find x and y.

Ans. $x=8$, *or* $-2\frac{1}{2}$; $y=2\frac{1}{2}$, *or* -8.

3. Given $2x+3y=8$ and $x^2+xy+y^2=7$, to find x and y.

SOLUTION.

$$2x+3y=8 \quad (1)$$
$$x^2+xy+y^2=7 \quad (2)$$

Assuming that $y=vx$, *and substituting this value of* y *in* (1) *and* (2), *we have,*

$$(2+3v)x=8 \quad (3)$$
$$(1+v+v^2)x^2=7 \quad (4)$$

Dividing (4) *by the square of* (3), *the* x^2 *is eliminated, and we have,*

$$\frac{1+v+v^2}{(2+3v)^2}=\frac{7}{64},$$

Whence, $v=2$, *or* 18

Substituting value of v *in* (3), $(2+6)x=8$

whence, $x=1$, *and* $y=2$

or, $(2+54)x=8$

whence, $x=\frac{1}{7}$, *and* $y=2\frac{4}{7}$

Here each of the equations is homogeneous, and one is simple and the other quadratic.

4. Given $x^2+xy=77$ and $xy-y^2=12$, to find x and y.

SOLUTION.

$$x^2+xy=77 \qquad (1)$$
$$xy-y^2=12 \qquad (2)$$

Assuming that $x=vy$, and substituting this value of x in (1) and (2), we have

$$v^2y^2+vy^2=77 \qquad (3)$$
$$vy^2-y^2=12 \qquad (4)$$

From (3), $$y^2=\frac{77}{v^2+v} \qquad (5)$$

From (4), $$y^2=\frac{12}{v-1} \qquad (6)$$

whence, $$\frac{77}{v^2+v}=\frac{12}{v-1} \qquad (7)$$

or, $$12v^2-65v=-77 \qquad (8)$$

Solving this equation, $$v=\frac{11}{3},\text{ or }\frac{7}{4},$$

Substituting these values in (6), *we have* $$y^2=\frac{12}{2\frac{2}{3}},\text{ or }\frac{12}{\frac{3}{4}}$$

or, $$y^2=\frac{9}{2},\text{ or }16$$

whence, $$y=\pm\frac{3}{\sqrt{2}}=\pm\frac{3}{2}\sqrt{2},\text{ or }\pm 4$$

and, $$x=vy=\pm\frac{11}{2}\sqrt{2},\text{ or }\pm 7$$

In this example each of the given equations is homogeneous and quadratic. It will be also seen that each of the unknown quantities has really *four* values, for $\sqrt{2}$ is either + or − (Art. 244, 2).

315. Rules for Solving some Simultaneous Quadratic Equations containing Two Unknown Quantities.

1. If one of the equations is simple and the other is quadratic, from the simple equation find an expression for the value of either of the unknown quantities, and substitute this value in the quadratic equation.

2. If each of the equations is homogeneous, substitute in both equations for one of the unknown quantities the product of v by the other.

In some cases either rule can be applied; the pupil then should use the rule which will solve the equations with the greater convenience.

1. Given $\left\{\begin{array}{l} x+y=10 \\ 3xy=72 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=6, \text{ or } 4, \\ y=4, \text{ or } 6. \end{array}\right.$

2. Given $\left\{\begin{array}{l} x-y=6 \\ x^2+y^2=50 \end{array}\right\}$, to find x and y.

3. Given $\left\{\begin{array}{l} x+4y=14 \\ 4x-2y+y^2=11 \end{array}\right\}$, to find x and y.
Ans. $\left\{\begin{array}{l} x=2, \text{ or } -46, \\ y=3, \text{ or } \quad 15. \end{array}\right.$

4. Given $\left\{\begin{array}{l} 4(x+y)=5 \\ 8xy=3 \end{array}\right\}$, to find x and y.

5. Given $\left\{\begin{array}{l} 2x-3y=0 \\ 2x^2+xy-5y^2=4 \end{array}\right\}$, to find x and y.
Ans. $\left\{\begin{array}{l} x=3, \text{ or } -3, \\ y=2, \text{ or } -2. \end{array}\right.$

6. Given $\left\{\begin{array}{l} x^2+xy+2y^2=44 \\ 2x^2-xy+y^2=16 \end{array}\right\}$, to find x and y.

7. Given $\left\{\begin{array}{l} x^2+xy=66 \\ xy-y^2=5 \end{array}\right\}$, to find x and y.
Ans. $\left\{\begin{array}{l} x=\pm 6, \text{ or } \pm \frac{11}{2}\sqrt{2}, \\ y=\pm 5, \text{ or } \pm \frac{1}{2}\sqrt{2}. \end{array}\right.$

8. Given $\left\{\begin{array}{l} 9x^2=4y^2 \\ 3xy+2x+y=485 \end{array}\right\}$, to find x and y.

9. Given $\left\{\begin{array}{l} x^2-xy-y^2=295 \\ x^2+y^2=370 \end{array}\right\}$, to find x and y.
Ans. $\left\{\begin{array}{l} x=\pm 19, \text{ or } \pm 7\sqrt{5}, \\ y=\pm 3, \text{ or } \mp 5\sqrt{5}. \end{array}\right.$

316. Some **Quadratics and Higher Equations,** which are symmetrical, may be readily solved by artifices, especially such as are suggested by the relations existing between the sum, difference and product of the unknown quantities.

1. Given $x^2+y^2=25$ and $xy=12$, to find x and y.

SOLUTION.

	$x^2+y^2=25$	(1)
	$xy=12$	(2)
Multiplying (2) *by* 2,	$2xy=24$	(3)
Adding (1) *and* (3),	$x^2+2xy+y^2=49$	(4)
or,	$(x+y)^2=49$	(5)
whence,	$x+y=\pm 7$	(6)
Subtracting (3) *from* (1),	$x^2-2xy+y^2=1$	
or,	$(x-y)^2=1$	(7)
whence,	$x-y=\pm 1$	(8)
From (6) *and* (8),	$x=\pm 3$, *or* ± 4	(9)
and,	$y=\pm 4$, *or* ± 3	(10)

Here, the equation, which can be solved by substitution, is more readily solved by an artifice which avoids the process of completing the square in the solution.

2. Given $\left\{\begin{matrix} xy=28 \\ x^2+y^2=65 \end{matrix}\right\}$, to find x and y.

Ans. $\left\{\begin{matrix} x=\pm 7, \text{ or } \pm 4, \\ y=\pm 4, \text{ or } \pm 7. \end{matrix}\right.$

3. Given $x^2-y^2=24$ and $x+y=6$, to find x and y.

SOLUTION.

	$x^2-y^2=24$	(1)
	$x+y=6$	(2)
Dividing (1) *by* (2),	$x-y=4$	(3)
Adding (3) *to* (2),	$2x=10$	(4)
whence,	$x=5$	(5)
Subtracting (3) *from* (2),	$2y=2$	
whence,	$y=1$	

4. Given $\left\{\begin{matrix} x^2+y^2=25 \\ x+y=1 \end{matrix}\right\}$, to find x and y.

5. Given $x+y=5$ and $x^3+y^3=65$, to find x and y.

SOLUTION.

	$x+y=5$	(1)
	$x^3+y^3=65$	(2)
Dividing (2) *by* (1),	$\dfrac{x^3+y^3}{x+y}=\dfrac{65}{5}$	(3)
or,	$x^2-xy+y^2=13$	(4)
Squaring (1),	$x^2+2xy+y^2=25$	(5)
Subtracting (4) *from* (5),	$3xy=12$	(6)
Dividing by 3,	$xy=4$	(7)
Multiplying by 4,	$4xy=16$	(8)
Subtracting (8) *from* (5),	$x^2-2xy+y^2=9$	(9)
Extracting the square root,	$x-y=\pm 3$	(10)
From (1) *and* (10),	$x=1$, *or* 4	(11)
and,	$y=4$, *or* 1	(12)

Here, after (4) was found, the solution could have been made also by combining (1) and (4), and applying rule 1.

6. Given $\begin{cases} x-y=2 \\ x^3-y^3=8 \end{cases}$, to find x and y. *Ans.* $\begin{cases} x=2, \text{ or } \ 0, \\ y=0, \text{ or } -2. \end{cases}$

7. Given $\begin{cases} x^2-y^2=9 \\ x^4-y^4=369 \end{cases}$, to find x and y. *Ans.* $\begin{cases} x=\pm 5, \\ y=\pm 4. \end{cases}$

SECTION XLI.

PROBLEMS PRODUCING SIMULTANEOUS QUADRATIC EQUATIONS.

317.—Ex. 1. Into what two parts can the number 40 be separated, so that the sum of their squares shall be 818?

Ans. 17 *and* 23.

2. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. What are the numbers?

3. The product of two numbers is 192, and the sum of their squares is 640. What are the numbers?

4. The sum of the squares of two numbers is 170, and the differenee of their squares is 72. What are the numbers?

5. A farmer having sold 7 lambs and 12 sheep for \$50, found that the number of lambs he had sold for \$10 was 3 more than the number of sheep he had sold for \$6. What was the price of each?

SOLUTION.

x = *the number of dollars each lamb sold for,*

y = *the number of dollars each sheep sold for;*

$\frac{10}{x}$ = *the number of lambs sold for \$10.*

$\frac{6}{y}$ = *the number of sheep sold for \$6.*

By conditions, $$\frac{10}{x}=\frac{6}{y}+3 \qquad (1)$$

and, $$7x+12y=50 \qquad (2)$$

From (2), $$x=\frac{50-12y}{7} \qquad (3)$$

Substituting this value of x in (1), $$70=(50-12y)\left(\frac{6}{y}+3\right)$$

whence, $$9y^2-2y=75$$

or, $$y^2-\frac{2}{9}y=\frac{75}{9}$$

Completing the square, $$y^2-\frac{2}{9}y+\left(\frac{1}{9}\right)^2=\frac{676}{81}$$

whence, $$y=3, \text{ or } -\frac{25}{9}$$

and, $$x=2, \text{ or } \frac{250}{21}$$

From the nature of the question the negative result is not admissible. The price of the lambs is \$2 each, and of the sheep, \$3.

6. What are the two numbers whose sum multiplied by the greater is 60, and whose difference multiplied by the less is 8?

7. There is a certain number expressed by two digits. The sum of the squares of the digits is equal to the number increased by the product of its digits, and if 36 be added to the number, the digits will be reversed. What is the number?

Ans. 37 or 48.

8. The difference of two numbers is 3, and the difference of their cubes is 279. What are the numbers?

9. The sum of two numbers is 20, and the sum of their cubes is 2240. What are the numbers? *Ans. 12 and 8.*

10. A and B invest in a certain partnership, having a joint capital of \$100. A leaves his money in the partnership for 3 months, and B leaves his for 2 months, and each at last realizes \$99 of capital and profit, the monthly rate of profit being uniform. What sum did each originally contribute?

SOLUTION.

x = *the number of dollars in A's capital,*

$100 - x$ = *the number of dollars in B's capital;*

y = *the monthly rate of gain.*

By conditions, $x + 3xy = 100 - x + 2y(100 - x)$ (1)

and, $x + 3xy = 99$ (2)

From (1), $y = \dfrac{100 - 2x}{5x - 200}$ (3)

Substituting this value in (2), $x^2 + 395x = 19800$

whence, $x = 45$

and, $100 - x = 55$

Hence, A's capital was \$45, and B's was \$55.

11. A grazier bought as many sheep as cost him \$300, and after reserving 15 out of the number, sold the remainder for \$270, and gained \$.50 a head. How many sheep did he buy?

12. Two traders jointly invest $500 in business. One of them let his money remain 5 months, the other only 2 months, and each received $450 capital and profit. How much did each invest? *Ans. One, $200; the other, $300.*

Test Questions.

318.—1. What is a *Quadratic Equation?* A pure equation? A pure quadratic equation? What are the given principles of quadratic equations? What is the rule for solving pure quadratic equations?

2. What is an *Affected Quadratic Equation?* What are the given principles of affected quadratic equations? What is the rule for solving an affected quadratic equation? Give the method which avoids fractions in the process of solution.

3. What is a *Homogeneous Equation?* A symmetrical equation? What cases of simultaneous quadratics can be treated in an elementary work? What are the rules for solving some simultaneous quadratic equations?

SECTION XLII.

RATIO AND PROPORTION.

319. Ratio is the relation which one of two similar quantities bears to the other, with respect to magnitude. It is ascertained by the division of the first of the given quantities by the second.

Thus, the ratio of a to b is $\frac{a}{b}$.

320. The **Sign** of ratio is the colon (:).

Thus, 5 : 6 denotes the ratio of 5 to 6, or $\frac{5}{6}$.

321. The **Terms** of a ratio are the two quantities whose magnitudes are compared.

The first term is called the **Antecedent,** and the second term is called the **Consequent.**

322. A **Simple Ratio** is a ratio each term of which is a single quantity.

A **Compound Ratio** is a ratio formed by multiplying together the corresponding terms of two or more simple ratios.

Thus, $a : b$ is a simple ratio; and $(a : b)(c : d)$, or $\frac{a}{b}\times\frac{c}{d}$, is a compound ratio.

323. A **Ratio of Equality** exists when the antecedent and consequent are equal.

A ratio of **greater inequality** exists when the antecedent is greater than the consequent; and of **less inequality,** when the antecedent is less than the consequent.

The ratio is said to be **direct** when the antecedent is divided by the consequent; and **inverse** or **reciprocal** when the consequent is divided by the antecedent.

324. Principles.—1. *Both terms of a ratio may be multiplied or divided by the same quantity without affecting its value.*

For, $a : b = \frac{a}{b}$; multiplying both terms of $\frac{a}{b}$ by n, we have $\frac{an}{bn}$, and dividing both terms of $\frac{an}{bn}$ by n, we have $\frac{a}{b}$, the value in each case being unchanged.

2. *A compound ratio is the ratio of the product of its antecedents to the product of its consequents.*

For, $(6 : 2)(5 : 3)$, or $\frac{6 : 5}{2 : 3}$, is equal to $\frac{6+5}{2\times 3}$.

PROBLEMS.

1. What is the ratio of 15 dimes to 3 dollars?

2. Which is the greater ratio, 3 : 4, or 2 : 3?

3. Find the ratio compounded of 4 : 15 and 5 : 6.

4. Express as a fraction the ratio of 5 to 17.

5. Which is the greater ratio, 5 : 6, or 123 : 148? *Ans. 5 : 6.*

6. Which is the greater ratio, 5 : 6, or $\frac{1}{5} : \frac{1}{6}$? *Ans.* $\frac{1}{5} : \frac{1}{6}$.

7. Two equal glasses are both full of mixtures of wine and water. In the one the ratio of the wine to the water is 1 : 7; in the other, the ratio is 1 : 9. When the two are poured into the same vessel, is the ratio of the wine to the water greater or less than 1 : 8? *Ans. Greater.*

PROPORTION.

325. **Proportion** is an equality of ratios.

Thus, $a : b = c : d$ expresses a proportion.

326. The **Sign** of proportion is a double colon (: :), which, instead of the sign of equality, may be placed between two equal ratios.

Thus, $a : b :: c : d$ denotes a proportion.

Each ratio is called a couplet, and each term a proportional.

327. The **Antecedents** of a proportion are the antecedents of its ratios.

The **Extremes** of a proportion are its first and fourth terms, and the means are its second and third terms.

328. A **Mean Proportional** of two quantities is a quantity that serves as both the means of a proportion in which the two quantities are the extremes.

Thus, b is a mean proportional in $a : b :: b : c$.

329. A **Continued Proportion** is a proportion in which each antecedent, except the first, is the same as the preceding consequent.

Thus, $a : b :: b : c :: c : d$ is a continued proportion.

330. Quantities are said to be in proportion by **alternation** when antecedent is compared with antecedent and consequeut with consequent; in proportion by **inversion** when the antecedents are made consequents and the consequents antecedents; in proportion by **composition** when the sum of antecedent and consequent is compared with either antecedent or consequent; and in proportion by **division** when the difference of antecedent and consequent is compared with antecedent or consequent.

Theorem I.

331. *In any proportion the product of the extremes is equal to the product of the means.*

Let $a : b :: c : d;$

then, $\frac{a}{b} = \frac{c}{d};$

Clearing of fractions, $ad = bc.$

Theorem II.

332. *If the product of two quantities be equal to the product of two other quantities, either two may be made the extremes of a proportion, and the other two the means.*

Let $ad = bc;$

Dividing by bd, $\frac{a}{b} = \frac{c}{d};$

or, $a : b :: c : d.$

Theorem III.

333. *In a proportion, either extreme is equal to the product of the means divided by the other extreme, and either mean is equal to the product of the extremes divided by the other mean.*

Let $a : b :: c : d.$

By Theorem I, $ad = bc;$

whence, by division, $a = \frac{bc}{d};\ d = \frac{bc}{a};\ b = \frac{ad}{c}$, and $c = \frac{ad}{b}.$

Theorem IV.

334. *The mean proportional between two quantities is equal to the square root of their product.*

Let $a : b :: b : c.$

By Theorem I, $b^2 = ac;$

Extracting square root, $b = \sqrt{ac}.$

Theorem V.

335. *If four quantities be in proportion, they will be in proportion by alternation.*

Let $a : b :: c : d.$

By Theorem I, $ad = bc.$

Dividing by dc, $\frac{a}{c} = \frac{b}{d};$

or, $a : c :: b : d.$

Theorem VI.

336. *If four quantities be in proportion, they will be in proportion by inversion.*

Let $a : b :: c : d.$

By Theorem I, $bc = ad;$

whence (Theorem II), $b : a :: d : c.$

Theorem VII.

337. *If four quantities be in proportion, the first together with the second will be to the second as the third together with the fourth is to the fourth.*

Let $a : b :: c : d;$

then, $\dfrac{a}{b} = \dfrac{c}{d}.$

Adding 1 to each member, $\dfrac{a}{b}+1=\dfrac{c}{d}+1;$

or, $\dfrac{a+b}{b}=\dfrac{c+d}{d};$

whence, $a+b : b :: c+d : d.$

Theorem VIII.

338. *If four quantities be in proportion, the excess of the first above the second will be to the second as the excess of the third above the fourth is to the fourth.*

Let $a : b :: c : d;$

then, $\dfrac{a}{b}=\dfrac{c}{d}.$

Subtracting 1 from each member, $\dfrac{a}{b}-1=\dfrac{c}{d}-1;$

or, $\dfrac{a-b}{b}=\dfrac{c-d}{d},$

whence, $a-b : b :: c-d : d.$

Theorem IX.

339. *If the corresponding terms of two or more proportions be multiplied together, the products will be proportionals.*

Let $a:b::c:d,$

and $e:f::g:h.$

Then, $\frac{a}{b}=\frac{c}{d};$

and $\frac{e}{f}=\frac{g}{h}.$

Multiplying, $\frac{ae}{bf}=\frac{cg}{dh};$

whence, $ae:bf::cg:dh.$

Theorem X.

340. *If four quantities be in proportion, like powers or roots of those quantities will be proportionals.*

Let $a:b::c:d;$

then, $\frac{a}{b}=\frac{c}{d}.$

Raising to the nth power, $\frac{a^n}{b^n}=\frac{c^n}{d^n}.$

Extracting the nth root, $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}};$

whence, $a^n:b^n::c^n:d^n;$

and, $a^{\frac{1}{n}}:b^{\frac{1}{n}}::c^{\frac{1}{n}}:d^{\frac{1}{n}}.$

PROBLEMS.

1. In $4:7::8:x$, find x. *Ans.* $x=14$.

2. In $5:x::x:45$, find x.

3. In $x+4:x+2::x+8:x+5$, to find x. *Ans.* $x=4$.

4. Find a fourth proportional to ab, cd and ax.

5. Find the number to which, if 2 and 5 be successively added, the resulting sums will be the ratio of $5:11$. *Ans.* $\frac{1}{2}$.

6. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$.

7. Find a mean proportional to 2 and 8.

8. Divide \$1000 between two persons so that their shares shall be in the ratio of 7 to 9.

SOLUTION.

$x =$ *number of dollars in first person's share;*

$1000 - x =$ *number of dollars in second person's share.*

By conditions,	$x : 1000 - x :: 7 : 9$
By Theorem I,	$9x = 7000 - 7x$
whence,	$x = 437.50$
and,	$1000 - x = 562.50$

Hence, the first person's share is \$437.50, and the second person's share, \$562.50.

9. Divide the number 56 into two parts, such that one shall be to the other as 3 to 4. *Ans.* 24; 32.

10. A and B are at present of the same age. If A's age be increased by 36 years, and B's by 52 years, their ages will be as 3 to 4. What is the present age of each?

11. A, B and C make a joint stock. A puts in \$60 less than B, and \$68 more than C; and the sum of the shares of A and B is to the sum of the shares of B and C as 5 to 4. What did each put in? *Ans. A, \$140; B, \$200; C, \$72.*

12. Find two numbers whose sum is to their difference as 3 to 2, and whose difference is to their product as 1 to 5.

13. A person in a railway-car observes that another train running on a parallel line in the opposite direction occupies 2 seconds in passing his train. But if the two trains had been proceeding in the same direction, one would have taken 30

seconds to pass the other. Compare the speed of the two trains.

SOLUTION.

$x : y =$ *the ratio of the rates.*

By conditions, $x-y : x+y :: 2 : 30$

whence, $x : y :: 8 : 7$

That is, the speed of the two trains is as 8 to 7.

14. There is a rectangular field which contains 360 square rods, and whose length is to its breadth as 8 to 5. What are the length and breadth?

Ans. Length, 24 rods; breadth, 15 rods.

15. In a court there are two square grass-plots, a side of one of which is 10 yards longer than a side of the other, and their areas are as 25 to 9. What are the lengths of their sides?

Ans. 15 yards; 25 yards.

SECTION XLIII.

ARITHMETICAL PROGRESSION.

341. A **Progression** is a succession or series of quantities increasing or decreasing according to some fixed law.

342. The **terms** of a progression, or series, are the quantities of which the series is formed.

343. An **Arithmetical Progression** is a series formed by adding a constant quantity.

The constant quantity added is called the **Common Difference,**

and the progression is **ascending** when the common difference is positive, and **descending** when it is negative.

Thus, $a, \quad a+d, \quad a+2d, \quad a+3d, \ldots$

is an ascending arithmetical progression in which the common difference is $+d$, and

$$a, \quad a-d, \quad a-2d, \quad a-3d, \ldots$$

is a descending arithmetical progression in which the common difference is $-d$.

344. In an **Arithmetical Progression** having a limited number of terms there are **five elements** to be considered:

1. The first term, a; ***3. The number of terms, n;***
2. The last term, l; ***4. The common difference, d;***
5. The sum of the terms, S.

and any three of these being known, the other two can be found.

The first and last terms are called **Extremes**, and all the other terms are called **Arithmetical Means.**

Theorem I.

345. *The last term of an arithmetical progression is equal to the first term plus the product of the common difference by the number of terms less one.*

Let the terms of the progression be

$$a, \quad a+d, \quad a+2d, \quad a+3d, \ \ldots$$

The coefficient of d in the last term is one less than the number of terms; hence, the nth term equals

$$a+(n-1)d,$$

$$\text{or,} \quad l=a+(n-1)d. \qquad (1)$$

Theorem II.

346. *The sum of all the terms of an arithmetical progression is equal to half the sum of the two extremes multiplied by the number of terms.*

Let an arithmetical progression be

$$a, \quad a+d, \quad a+2d, \quad a+3d, \ldots l$$

Then, since the sum of a series is the same, whether written in the direct or in the inverse order,

$$\begin{array}{rl} & S=a+(a+d)+(a+2d)+(a+3d)+\ldots+l \\ & S=l+(l-d)+(l-2d)+(l-3d)+\ldots+a \\ \hline \textit{Adding,} & 2S=(a+l)+(a+l)+(a+l)+(a+l)+\ldots+(a+l). \end{array}$$

Here, $a+l$ is taken as many times as there are terms, or n times.

$$\text{Hence, } 2S=n(a+l)\text{; whence, } S=\left(\frac{a+l}{2}\right)n \qquad (2)$$

Theorem III.

347. *Any number of arithmetical means may be inserted between two given terms of an arithmetical progression.*

The number of terms in a series must consist of the two extremes and all the intermediate terms, or of two more terms than the number of means. Hence, if m be made to denote the number of means,

$$m+2=n,$$

or the whole number of terms.

Substituting the value of n in formula (1), we have

$$l=a+(m+1)d, \qquad (3)$$

$$\textit{whence,} \quad d=\frac{l-a}{m+1}. \qquad (4)$$

Hence, the required means may be obtained by the continued addition of the value of d.

Theorem IV.

348. *An arithmetical mean between any two quantities is one-half of their sum.*

For, when $m=1$, formula (4) becomes

$$d=\frac{l-a}{2} \qquad (5)$$

Adding a to each member, and reducing, we have

$$a+d=\frac{l+a}{2} \qquad (6)$$

But $a+d$ is the second term of a series whose first term is a and

whose common difference is d: that is, the arithmetical mean of the series $a, \quad a+d, \quad a+2d$.

349. The two formulas,

$$l=a+(n-1)d, \tag{1}$$

$$S=\left(\frac{a+l}{2}\right)n, \tag{2}$$

are fundamental, since from them may be derived other formulas, by which, when any three of the five elements of an arithmetical progression are given, the other two can be determined. Thus, we may obtain, among others:

1. For finding the first term:

$$a=l-(n-1)d, \tag{3}$$

$$a=\frac{2S}{n}-l, \tag{4}$$

$$a=\frac{S}{n}-\frac{(n-1)d}{2}. \tag{5}$$

2. For finding the last term:

$$l=\frac{2S}{n}-a, \tag{6}$$

$$l=\frac{S}{n}+\frac{(n-1)d}{2}. \tag{7}$$

3. For finding the common difference:

$$d=\frac{l-a}{n-1}, \tag{8}$$

$$d=\frac{2(S-an)}{n(n-1)}, \tag{9}$$

$$d=\frac{2(nl-S)}{n(n-1)}, \tag{10}$$

4. For finding the number of terms:

$$n=\frac{l-a}{d}+1, \tag{11}$$

$$n=\frac{2S}{a+l}. \tag{12}$$

5. For finding the sum of the terms:

$$S=\frac{n}{2}[2a+(n-1)d], \tag{13}$$

$$S=\frac{(l+a)(l-a+d)}{2d}. \tag{14}$$

PROBLEMS.

1. Given $a=7$, $d=3$ and $n=9$, to find l. *Ans.* $l=31$.

2. Given $a=10$, $d=-3$ and $n=4$, to find l.

3. Given $a=7$, $l=31$ and $n=9$, to find S. *Ans.* $S=171$.

4. Given $a=10$, $l=70$ and $n=21$, to find d.

5. Given $l=31$, $d=3$ and $n=9$, to find a. *Ans.* $a=7$.

6. Given $a=4$, $l=11$ and $n=10$, to find d.

7. Given $S=171$, $a=7$ and $l=31$, to find n. *Ans.* $n=9$.

8. Find the sum of the series 2, 6, 10, 14 . . . to 20 terms.

9. Insert five arithmetical means between 11 and 23.

SOLUTION. Here, $l=23$, $a=11$ and $m=5$; whence, by formula (4), Art. 347, $d=2$. Hence, by addition, the progression will be 11, 13, 15, 17, 19, 21, 23.

$$d=\frac{23-11}{5+1}=2$$

10. Find an arithmetical mean between 12 and 20. *Ans. 16.*

11. Form an arithmetical progression by inserting 4 arithmetical means between 2 and —18.

12. How many strokes does a common clock make in 12 hours, if it strikes the half hours? *Ans. 90.*

13. A debt can be discharged in a year by paying \$1 the first week, \$3 the second, \$5 the third, and so on. Required the last payment and the amount of the debt.

14. A person has a journey of 140 miles to perform. He goes 26 miles the first day, 24 miles the second, 22 the third, and so on. In how many days does he perform the journey?

SECTION XLIV.

GEOMETRICAL PROGRESSION.

350. A **Geometrical Progression** is a series in which each term, after the first, is equal to the preceding term multiplied by a constant factor.

351. The **Rate** or **Ratio** is the constant factor. The progression is **ascending** when the rate is greater than 1, and **descending** when the rate is less than 1.

Thus, a, ar, ar^2, ar^3 ...... is a geometrical progression in which r is the rate, and which is ascending or descending, according as r is greater or less than 1.

If the rate is a negative quantity, the terms of the progression will be alternately positive and negative, or negative and positive, according as the first term is positive or negative.

Thus, if the rate is -2, and the first term is 5, the progression will be,—

$$5, \quad -10, \quad 20, \quad -40, \quad 80 \ldots;$$

and if the first term is -5, the progression will be,—

$$-5, \quad +10, \quad -20, \quad +40, \quad -80 \ldots$$

352. An **Infinite Series** is a descending progression of an infinite number of terms.

The last term of such a series must be less than any assignable quantity; hence, it may be considered to be 0.

Thus, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ .... 0 is an infinite series.

353. The **Sum** of an infinite series, or the sum of the series to infinity, is the limit which the sum approaches as the number of terms increases.

The sum of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ ... is constantly approaching the limit 2, and can never go beyond it. Hence, if the series be continued to infinity, its sum will be 2.

354. In a **Geometrical Progression** there are five elements to be considered:

1. The first term, a; *3. The number of terms, n;*
2. The last term, l; *4. The rate or ratio, r;*
5. The sum of the terms, S.

The first and last terms are called **Extremes**, and the other terms are called **Geometrical Means.**

Theorem I.

355. *The last term of a geometrical progression is equal to the product of the first term by that power of the rate whose exponent is one less than the number of terms.*

Let the terms of the progression be $a, \quad ar, \quad ar^2, \quad ar^3 \ldots$

The exponent of r in the last term is one less than the number of terms; hence, the nth term equals ar^{n-1}

$$Or, \qquad l = ar^{n-1} \qquad (1)$$

Theorem II.

356. *The sum of all the terms of a geometrical progression is equal to the difference between the first term and the product of the last term, multiplied by the rate, divided by the rate less one.*

Let a geometrical progression be $a, \quad ar, \quad ar^2, \quad ar^3 \ldots$

Then, $S = a + ar + ar^2 + ar^3 + \ldots + l$

Multiplying by r, $rS = \quad ar + ar^2 + ar^3 + \ldots + l + rl$

Subtracting, $rS - S = rl - a$

or, $(r-1)S = rl - a$

whence, $$S = \frac{rl - a}{r - 1} \qquad (2)$$

Theorem III.

357. *The sum of an infinite series is equal to the quotient of the first term divided by one minus the rate.*

The progression being descending, formula (2), in order that the denominator may be positive, must become,

$$S=\frac{a-rl}{1-r} \tag{3}$$

Now, since the number of terms in the descending series is infinite, the last term may be considered 0, and the last term multiplied by the rate will also be 0. Hence, the formula (3) will become,

$$S=\frac{a-0}{1-r}$$

$$\text{or,}\qquad S=\frac{a}{1-r} \tag{4}$$

Theorem IV.

358. *Any number of geometrical means may be inserted between two given terms of a geometrical progression.*

The number of terms in a series must consist of two more terms than the number of means. Hence, if m be made to denote the number of means,

$$m+2=n,$$

the whole number of terms. Substituting this value of n in formula (1), Art. 355, we have,

$$l=ar^{m+1} \tag{5}$$

$$\textit{whence,}\qquad r=\sqrt[m+1]{\frac{l}{a}} \tag{6}$$

Having found the rate, we may obtain the required means by successively multiplying the first term by the rate, by its square, by its cube, etc.

Theorem V.

359. *The geometrical mean between two quantities is the square root of their prodact.*

When $m=1$, formula (6) gives $r=\sqrt{\frac{l}{a}}$ (7)

Multiplying by a and reducing $ar=\sqrt{al}$ (8)

But ar is the second term of a series whose first term is a, and whose rate is r; that is, the geometrical mean of the series a, ar, ar^2.

360. The two formulas,

$$l = ar^{n-1} \tag{1}$$

$$S = \frac{rl - a}{r - 1} \tag{2}$$

are fundamental, since from them may be derived other formulas, by which, when any three of the five elements of a geometrical progression are given, the others may be determined. Thus, we may obtain, among others:

1. For finding the first term:

$$= \frac{l}{r^{n-1}}$$

$$a = \frac{(r - l)S}{r^n - 1}$$

$$a = rl - (r - 1)S$$

2. For finding the last term:

$$l = \frac{a + (r - 1)S}{r}$$

$$l = \frac{(r - 1)Sr^{n-1}}{r^n - 1}$$

3. For finding the rate:

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \sqrt[n-1]{\frac{l}{a}}$$

$$r = \frac{S - a}{S - l}$$

4. For finding the sum:

$$S = \frac{ar^n - a}{r - 1}$$

$$S = \frac{r^n l - l}{r^n - r^{n-1}} = \frac{l(r^n - 1)}{(r - 1)r^{n-1}}$$

5. For finding the number of terms:

$$n = \frac{\log. l - \log. a}{\log. r} + 1$$

$$n = \frac{\log. l - \log. a}{\log.(S - a) - \log.(S - l)} + 1$$

The finding of the value of n involves a knowledge of logarithms, but the formulas are given here for convenience of reference.

PROBLEMS.

1. Given $a=2$, $r=2$, and $n=8$, to find l. *Ans.* $l=256$.

2. Given $l=256$, $r=2$, and $n=8$, to find a.

3. Given $a=2$, $l=54$, and $r=3$, to find S. *Ans.* $S=80$.

4. Given $a=4$, $l=12500$, and $S=15624$, to find r.

5. Given $a=4$, $r=5$, and $n=6$, to find S.

6. What is the sum of the series 1, 4, 16, etc., to 6 terms?
Ans. 1365.

7. What is the sum of the series, 1, $\frac{1}{4}$, $\frac{1}{16}$, etc., to infinity?

8. What is the value of $25+10+4+$, etc., to 4 terms?
Ans. $40\frac{3}{5}$.

9. What is the value of $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+$, etc., to 8 terms?

10. Find the sum of the series $\frac{1}{3}$, $\frac{2}{9}$, $\frac{4}{27}$, etc., to infinity.
Ans. 1.

11. Find a geometrical mean between 20 and $31\frac{1}{4}$.

12. Insert two geometrical means between 5 and 135.
Ans. 15 and 45.

13. Required the value of $.\dot{4}\dot{5}$.

SOLUTION.

$$.\dot{4}\dot{5}=\frac{45}{100}+\frac{45}{10000}+\frac{45}{1000000}\cdots$$

or the sum of an infinite series, of which the first term is .45 and the rate .01. Then, by formula (4), Art. 357,—

$$.\dot{4}\dot{5}=.45\div(1-.01)=\frac{45}{99}=\frac{5}{11}.$$

14. What is the value of .162162... to infinity? *Ans.* $\frac{18}{111}$.

15. A man who had been engaged in speculation 7 years found that his capital had been trebled each year, and that, as the result, he then had \$109,350. How much money did he have at first?

16. A debt of \$4095 can be discharged in 12 monthly payments, which increase in geometrical progression. If the first payment is \$1 and the last \$2048, what is the rate? *Ans. 2.*

17. How far would a man travel in 5 days if he should go 40 miles the first day, 30 miles the second, and so on, going each day three-fourths as far as on the preceding day?

18. A man bought successively 7 horses, giving 3 times as much for each as for the last preceding one, the entire cost being \$2186. What was the price of the first horse, and of the last?

Ans. The first, \$2; the last, \$1458.

SECTION XLV.

PROBLEMS SOLVED BY APPLICATION OF PRINCIPLES OF THE PROGRESSIONS.

361.—Ex. 1. If you should save \$100 a year, and invest it at 6 per cent. simple interest, what will be the amount of your savings and interest at the end of 30 years?

SOLUTION.

Let $\$100 =$ *the first term of an arithmetical progression;*

then, $\$100 \times .06 = \$6 =$ *the com. dif. of an arithmetical progression;*

and, $30 =$ *the number of terms of an arith. progression.*

Then, by (1), *Art.* 345, $l = \$100 + (30-1)\$6 = \$274,$

and by (2), *Art.* 346, $S = \frac{30}{2}(\$100 + \$274) = \$5610,$

which is the amount required.

1' *

2. What uniform sum must a person save each year and put at 6 per cent. interest to amount to \$5610 in 30 years?

Ans. \$100.

3. A man who began the use of tobacco at the age of 20, at a cost of \$25 a year, found, after a number of years, that if he had put that sum each year in a savings bank, at 5 per cent. interest, he would have had \$1975. How many years had he used tobacco?

4. There are four numbers in arithmetical progression, the product of whose extremes is 45, and the product of whose means is 77. What are the numbers?

SOLUTION.

Let $x =$ *the first term;*

and $y =$ *the common difference;*

then, $x,\ x+y,\ x+2y,\ x+3y =$ *the arithmetical progression.*

By conditions, $x(x+3y)=x^2+3xy=45$ (1)

and, $(x+y)(x+2y)=x^2+3xy+2y^2=77$ (2)

Subtracting (1) *from* (2), $2y^2=32$ (3)

whence, $y=4$ (4)

Substituting the value of y *in* (1), $x^2+12x=45$ (5)

whence, $x=3$ (6)

Hence, the numbers are, 3, 7, 11, 15.

5. The sum of the squares of the first and the last of four numbers in arithmetical progression is 200, and the sum of the squares of the second and third is 136. Find the numbers.

6. The prices of three books are in arithmetical progression, and are such that their sum is \$15, and the price of the third, with double that of the second, is \$18. What are the prices?

7. The sum of four numbers in arithmetical progression is 34, and the sum of their squares 334. What are the numbers?

(Let $x+3y,\ x+y,\ x-y,\ x-3y$ represent the four terms.)

8. Find three numbers in geometrical progression whose sum is 14, and the sum of whose squares is 84.

SOLUTION.

Let $x =$ *one extreme;*

and $y =$ *the other extreme;*

then, $\sqrt{xy} =$ *the mean.*

By conditions,	$x+\sqrt{xy}+y=14$	(1)
and,	$x^2+xy+y^2=84$	(2)
Dividing (2) *by* (1),	$x-\sqrt{xy}+y=\ 6$	(3)
Adding (3) *and* (1),	$x+y=10$	(4)
Subtracting (3) *from* (4),	$\sqrt{xy}=\ 4$	(5)
Squaring (5),	$xy=16$	(6)
Subtracting 4 *times* (6) *from the square of* (4), *and taking square root of remainder,*	$x-y=6$	(7)
Subtracting (7) *from* (4),	$2y=4$	(8)
whence,	$y=2$	(9)
Substituting for y *in* (7),	$x=8$	(10)
and,	$\sqrt{xy}=4$	(11)

Hence, the numbers are 2, 4 and 8.

9. The sum of three numbers in geometrical progression is 26, and the sum of their squares is 364. What are the numbers? *Ans. 2, 6, 18.*

10. Of three numbers in geometrical progression the sum of the first two is 15, and the sum of the first and last is 25. What are the numbers?

11. To what sum will \$500 amount, at 7 per cent. compound interest, in 5 years? *Ans. \$701.27.*

Here, we have the first term of a geometrical progression, \$500; the rate, 1.07; and the number of terms, 6, to find the last term.

12. If \$50 should be saved and deposited in a savings bank every 6 months, at 3 per cent. semi-annual compound interest, what will be the amount of the deposits and interest at the end of 25 years?

13. If a man owes \$3000, what equal annual payments will in 5 years discharge both principal and interest, at 7 per cent. compound interest?

SOLUTION.

Let $p = \$3000$

and $r = 1.07$;

also, $n = 5$, *the number of years,*

and $x =$ *the annual payment.*

Then, the present worths of n *payments are expressed by*

$$\frac{x}{r}, \quad \frac{x}{r^2}, \quad \frac{x}{r^3}, \ldots \frac{x}{r^n}.$$

Since the sum of their present worths must equal the debt,

$$\frac{x}{r} + \frac{x}{r^2} + \frac{x}{r^3} \ldots + \frac{x}{r^n} = p.$$

whence, $$x = \frac{p}{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \ldots + \frac{1}{r^n}}$$

but, $$\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \ldots + \frac{1}{r^n} = \frac{r^n - 1}{r^n(r-1)};$$

hence, $$x = \frac{pr^n(r-1)}{r^n - 1} = \frac{\$3000 \times 1.07^5 \times (1.07 - 1)}{1.07^5 - 1}$$

or, $x = \$731.67.$

14. A farmer owes on his farm \$1000; what equal annual payments will in 10 years discharge the debt, at 6 per cent. compound interest? *Ans.* \$135.87.

15. Suppose the debt of a certain city to be \$18,000,000; what equal annual payments will in 25 years discharge it, at 6 per cent. compound interest?

16. If a man travel 50 miles the first day, and 45 miles the second, and so continue to travel 5 miles less each day, how far will he have gone on his journey at the end of the eighth day?

Test Questions.

362.—1. What is *Ratio?* The sign of ratio? What are the terms of a ratio? What is a simple ratio? A compound ratio? A ratio of equality? Of greater inequality? Of less inequality? What are the given principles of ratio?

2. What is *Proportion?* The sign of proportion? What are the antecedents of a proportion? The consequents? The extremes? The means? What is a mean proportional of two quantities? What is a continued proportion? What are the given theorems relating to proportion?

3. What is a *Series* or *Progression?* An arithmetical progression? The common difference? What are the given theorems relating to arithmetical progression?

4. What is a *Geometrical Progression?* The rate or ratio of a geometrical progression? An infinite series? The sum of an infinite series? What are the given theorems relating to geometrical progression?

SECTION XLVI.

GENERAL REVIEW.

363.—Ex. 1. Express in algebraic form the sum of x, y and z, divided by the product of x and the square of y.

2. Combine in one sum $3x^2y^2-10y^4$, $-x^2y^2+5y^4$, $8x^2y^2-6y^4$ and $x^2y^2+2y^4$.

3. Reduce to its simplest form the expression $5a-4b+3c+(-3a+2b-c)$. *Ans.* $2a-2b+2c$.

4. Multiply $2x-y$ by $2x-y$.

5. Prove that the continued product of $x-3$, $x+3$, $x-4$ and $x+4$ is x^4-25x^2+144.

6. Divide $a^4 - b^4$ by $a+b$.

7. Divide $5x^3y^3 - 40^2x^2y^2 + 25a^4xy$ by $-5xy$.

8. Divide $x^4 - y^4$ by $x-y$. *Ans.* $x^3 + x^2y + xy^2 + y^3$.

9. Expand $(-5ab^2x^3y^4)(-5ab^2x^3y^4)$.

10. Find the square of $6a^2 + 2a^2b$.

11. Factor $x^2 - 2xy + y^2$. *Ans.* $(x-y)(x-y)$.

12. What is the greatest common divisor of $8a^2xy$, $-12bxy^2$ and $20cx^2y$?

13. Show that the greatest common divisor of $10ax$ and $15x^2$ is $5x$.

14. Show that the least common multiple of a^2bc and $2ab^2d$ is $2a^2b^2cd$.

15. State in algebraic form that the product of a, b and c, divided by the difference of c and d, is equal to the cube of a divided by the sum of b and the square of c.

16. What is the least common multiple of a, $c+b$ and $c-b$?

17. Express algebraically a quantity with a numerical coefficient and a literal exponent.

18. What is the value of $4a^3 + (b^2 - c)(a^2 - c^2)$, a being equal to 3, b to 4 and c to 2?

19. What is the greatest common factor of $a^2 + 2ab + b^2$ and $a+b$? *Ans.* $a+b$.

20. Reduce $\dfrac{x^2 - 1}{xy + y}$ to its lowest terms.

21. What is the least common denominator of $\frac{a}{2b}$, $\frac{3b}{a}$, $\frac{5 \times y}{c}$?
Ans. $2abc$.

22. What is the sum of $\frac{a}{a+x}$ and $\frac{x}{a-x}$?

23. What single fraction expresses the sum of $\frac{a}{b}$, $\frac{c}{d}$ and $-\frac{x}{y}$?

24. Divide $\frac{9x^3-3x^2}{5x}$ by $\frac{x^2}{5}$. *Ans.* $\frac{9x-3}{x}$.

25. What is the product of $\frac{a^2x-x^3}{a}$ by $\frac{3a}{2ax-2x^2}$?

26. Multiply $\frac{x+y}{x-y}$ by $\frac{3x}{x+y}$. *Ans.* $\frac{3x}{x-y}$.

27. Divide $\frac{4x}{x^3-1}$ by $\frac{2}{x+y}$.

28. What is the value of x in $2x-\frac{19}{4}=\frac{3x}{4}+4$?

29. Show that the value of x in $\frac{3x-1}{7}+\frac{11-4x}{3}=\frac{12}{7}$, is 2.

30. Divide 21 into two parts, so that ten times one of them shall exceed nine times the other by 1.

31. A says to B, "You are 4 years more than twice as old as I am; 20 years ago you were 3 times as old as I was." What is now the age of each?

32. Solve the equation, $7x-\frac{4x-3}{2}+5\frac{1}{2}=8x+6\frac{1}{2}$.

33. Solve the equation, $mx+a=nx+d$. *Ans.* $x=\frac{d-a}{m-n}$.

34. Solve the equation $(m+n)(m-x)=m(n-x)$.

35. Solve the equation $\frac{9x+7}{2}=x-\frac{x-2}{7}+36$.

36. Solve the equation $ax+b^2=a^2+bx$.

37. What number is that to which if its third and its fourth be added, the sum will exceed its sixth by 17?

38. A man being asked the price of his watch, replied: "If you multiply the price by 4, and to the product add \$70, and from this sum subtract \$50, the remainder will be equal to \$220." How much did he give for his watch?

39. Two persons received equal sums of money; the first paid away \$25, and the second \$60. It then appeared that the former had twice as much as the latter. What sum did each receive? *Ans.* *\$95.*

40. A certain sum of money at interest amounts to \$297.80 in 8 months, and in 15 months to \$306. What is the sum, and what is the rate of interest?

41. A person bought two casks of vinegar, one of which held just 3 times as much as the other. From each of them he drew 4 gallons, and then found that there were 4 times as many gallons remaining in the larger as in the other. How many gallons were there in each?

Ans. *12 in one; 36 in the other.*

42. Find the value of x and y in the equations $x+y=36$ and $x-y=12$.

43. Given $4x+5y=32$ and $x+3y=15$, to find x and y.

44. Given $\frac{4x+3y}{6}=8$ and $\frac{7y-3x}{2}-y=11$, to find x and y.

45. Given $x=\frac{1}{7}(y-2)+5$ and $4y-\frac{1}{8}(x+10)=3$, to find x and y. *Ans.* $x=5$; $y=2$.

46. If a privateer discovers a ship, 18 miles off, sailing away at the rate of 8 miles an hour, and pursues her at the rate of 10 miles an hour, how long will the chase last?

47. Coffee and sugar were formerly cheaper than now, coffee by 2 cents and sugar by 4 cents a pound; and the price of the former was double that of the latter. The prices now are as 3 to 2; at what rate are they sold?

48. A steamboat whose speed in still water is 10 miles an hour, descends a river whose velocity is 4 miles an hour, and returns in 10 hours. How far did she proceed? *Ans. 42 miles.*

49. A farm, consisting of pasture and arable land, is hired at an annual rent of \$390, the pasture being valued at \$1.50 per acre, and the arable land at \$3. Now the number of acres of arable land is to half the excess of the arable above the pasture as 5 to 1. Required the quantity of each.

50. A farmer, disposing of his stock, sells to one person 9 horses and 7 cows for \$1200; and to another, 6 horses and 13 cows for the same sum. What was the price of each?

Ans. Cows, \$48; horses, \$96.

51. What fraction is that which, by the subtraction of 4 from both of its terms, becomes $\frac{1}{8}$, and by the addition of 9 to both, becomes $\frac{2}{3}$?

52. A rectangular lot of land was sold at the rate of \$2.25 per foot, for \$5400. Had its length been 10 feet less, its value would have been \$4725. What were its length and breadth? *Ans. Length, 80 feet; breadth, 30 feet.*

53. Three persons, A, B and C, have certain sums of money, such that A and B together have \$120; A and C together have \$140; and B and C together have \$150. What sum has each?

54. Given $5x+3y=65$, $2y-z=11$ and $3x+4z=57$, to find x, y and z. *Ans.* $x=7$; $y=10$; $z=9$.

19

55. Given $x+2y+3z=17$, $2x+3y+z=12$ and $3x+y+2z=13$, to find x, y and z.

56. Three persons, A, B and C, were talking of their ages. A said the sum of their ages was 90 years. B replied, that if his age were taken from the sum of the other two, the remainder would be 30 years. C said, if his age were taken from the sum of the other two, the remainder would be $\frac{1}{4}$ of his age. Required their ages.

57. What is the mth power of xy^pz^q? *Ans.* $x^my^{mp}z^{mq}$.

58. What is the cube of a^2+2a-4?
Ans. $a^6+6a^5-40a^3+96a-64$.

59. What is the square root of $x^2y^4z^6$?

60. What is the cube root of $-8a^3b^6x^9$? *Ans.* $-2ab^2x^3$.

61. What is the square root of $a^2x^2-2abxy^2+b^2y^4$?

62. What is the square root of $x^4-2x^3+3x^2-2x+1$?

63. Reduce $\sqrt{80a^3x^4}$ to its simplest form. *Ans.* $4ax^2\sqrt{5a}$.

64. What is the sum of $\sqrt{4ax^2}$ and $3x\sqrt{9a}$?

65. Express $-7a^2b$ in the form of the square root.
Ans. $\sqrt{49a^4b^2}$.

66. What is the sum of $3x\sqrt{9a}$ and $-7x\sqrt{a}$?

67. What factor will rationalize $\sqrt{a}+\sqrt{b}$? *Ans.* $\sqrt{a}-\sqrt{b}$

68. Solve the equation $3x^2+7=\dfrac{5x^2}{4}+35$. *Ans.* $x=\pm 4$.

69. Solve the equation $3x^2-29=\dfrac{x^2}{4}+510$.

70. Find two numbers whose sum is to the less as 10 to 3, and the difference of whose squares is 640.

71. A number of boys set out to get some apples, each carrying as many bags as there were boys, and each bag being capable of holding 6 times as many apples as there were bags. After filling their bags, they found the whole number of apples was 1536. How many boys were there?

72. If A's money were increased by half of B's, it would amount to \$54; and if B's remaining sum were trebled, it would exceed three times the difference of their original sums by \$6. How many dollars had each at first?

Ans. A, \$40; B, \$28.

73. The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards, and the circumference of one is a yard less than that of the other. Find the circumference of each.

74. A person changed a half-eagle for 25 pieces of foreign coins, some of which were reckoned as 30 to the half-eagle, and the others 15. How many did he get of each?

Ans. 20 of one kind; 5 of the other.

75. A market-woman having bought some eggs at 2 for 1 cent, and as many more at 3 for 1 cent, sold them at 5 for 2 cents and lost 4 cents by the transaction. How many cents did she pay out? *Ans. 100.*

76. A farmer has two cubical stacks of hay: the side of one is 3 yards longer than the side of the other, and the difference of their contents is 117 solid yards. Required the side of each. *Ans. 5 yards; 2 yards.*

77. Divide 100 into two parts, so that one shall be a multiple of 7, and the other a multiple of 11.

78. Two-thirds of a certain number of poor persons received

\$3 each, and the rest \$5 each, the whole sum spent being \$110. How many poor persons were there? *Ans. 30.*

79. Two girls together carried 25 eggs to market. They sold them at different prices; but each received the same amount. The one would have sold them all for 12 cents, and the other for 13 cents. How many did each sell?

80. A and B together carried 100 eggs to market, and each received the same sum. If A had carried as many as B, he would have received 18 cents more for them, and if B had carried as many as A, he would have received only 8 cents. How many had each?

81. Solve the equation $x^2-12x+20=0$. *Ans.* $x=10$, *and* 2.

82. Solve the equation $2x=4+\frac{6}{x}$.

83. Solve the equation $\frac{1}{3}x-\frac{1}{2}x=9$.

84. Solve the equation $x+\frac{24}{x-1}=3x-4$.

85. The expenses of a party are \$10, and if each pays 30 cents more than there are persons, the bill will be settled. How many persons are there in the party?

86. Find two numbers whose product is 100, and the difference of whose square roots is 3. *Ans. 25; 4.*

87. I bought a number of pieces of cloth for \$33.75, which were sold again at \$2.40 a piece, and as much was gained by the transaction as one piece cost. Find the number of pieces.

88. A person bought cloth for \$60. If he had bought 1 yard less for the same money, each yard would have cost \$.25 more. How many yards did he buy?

89. A grazier bought a certain number of oxen for \$240, and after losing 3 he sold the remainder for \$8 a head more than they cost him, thus gaining \$59 by his bargain. What number did he buy? *Ans. 16.*

90. A and B begin trade, A with 3 times as much stock as B. They each gain $50, and then 3 times A's stock is exactly equal to 7 times B's. What were their original stocks?

91. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he would have been 6 hours longer in performing the journey. Find his rate of travelling. *Ans. 7 miles an hour.*

92. A party at a tavern had a bill of $20 to pay, but two of the party having gone off, those who remained had each $.50 additional to pay. How many were there at first?

93. Insert two arithmetical means between 1 and 3.

94. A and B could do a piece of work in 4 days. A works alone for 2 days, and then they finish it in $2\frac{1}{2}$ days more. In what time could they have done it separately?

Ans. A, in $5\frac{1}{3}$ days; B, in 16 days.

95. A and B engaged in trade, A with $275 and B with $300. A lost half as much again as B, and B had then remaining half as much again as A. How much did each lose?

Ans. A, $135; B, $90.

96. A regiment of 1000 men is drawn up in an oblong form, so that the difference of the numbers in the two unequal sides is 117. Required the numbers in rank and file.

97. A gentleman pays $180 more for his carriage than for his horse; and the price of the carriage is to that of the horse as the latter is to 15. What is the price of each?

Ans. Carriage, $240; horse, $60.

98. A father left $210 to three sons, to be divided in sums that form a geometrical progression, so that the first should have $90 more than the last. What were their legacies?

99. Find two numbers, whose sum multiplied by the greater produces 130, and whose difference multiplied by the less gives 21.

100. A person took \$6 with him to distribute equally among some poor persons; but, as 5 of them were absent, the remainder received 10 cents apiece more than they otherwise would have received. How many poor persons were there?

101. A merchant bought a piece of cloth for \$40, and after cutting off 4 yards, sold the remainder at \$2.50 per yard, and received what the whole cost him. How many yards were there, and what did they cost? *Ans. 20 yards, at \$2.*

102. Find the greatest common divisor of $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ and $a^3 + x^3 - ax^2 - a^2x$. *Ans.* $a^2 - 2ax + x^2$.

103. A broker bought two cabinets at a sale, having been told that a \$10 gold piece was hidden in one of the two. If in the one, he thought it should be worth twice the other; but if in the other, it ought to be worth three times the first. What value did he put upon the cabinets without the gold piece?

104. Multiply $a^q + b^{st} + c^{-\frac{1}{q}}$ by $a^{-2q} + c^{\frac{1}{q}}$.

105. Two boats were sent out from a ship on a whaling excursion, with crews in the ratio of 3 to 5. Meeting some time after, they considered it would be better to divide the hands equally between the two boats, and did so by removing 6 men from one boat to the other. What was the whole number of men sent out? *Ans. 48.*

106. Find the value of x in the equation $\sqrt{x+a} = 2\sqrt{x+b}$.

107. A party of laborers were sent to remove a bank of earth containing 350 cubic yards; but just as they were commencing, four of them were disabled by an accident, in consequence of which each of the rest had ten additional yards of earth to remove. What was the number of the party?

Ans. 14.

108. A traveller, having proceeded 175 miles by railway, complained of the slowness of the train, and showed that, if it

had only gone five miles per hour faster, he would have accomplished his journey in less time by an hour and three-quarters. What was the rate at which the train went?

Ans. *20 miles per hour.*

109. Find the least common multiple of $(x+2a)^3$, $(x-2a)^3$ and x^2+4a^2.

110. Find two numbers such that, if 6 be added to each, they shall be to each other as 4 to 5, and if 4 be taken from each, they shall be to each other as 2 to 3. *Ans.* *14; 19.*

111. Given $\frac{1}{x}+\frac{1}{y}=a$, $\frac{1}{x}+\frac{1}{z}=b$, and $\frac{1}{y}+\frac{1}{z}=c$, to find x, y and z.

112. Given $x-\sqrt{x}=1$, to find x. *Ans.* $x=\frac{1}{2}(3\pm\sqrt{5})$.

113. A gentleman bought separately two contiguous fields containing together 40 acres. Each of them cost as many dollars per acre as there were acres in the field, and the prices of the two were in the proportion of 4 to 9. What were the areas of the two fields?

114. A person rents a certain number of acres of pasture land for \$70. He keeps 8 acres in his own possession, sublets the remainder at 25 cents per acre more than he gave, and thus covers his rent and \$2 over. What was the number of acres?

115. A and B are two towns situated 18 miles apart on the same bank of a river. A man goes from A to B in 4 hours, by rowing the first half of the distance and walking the second half. In returning, he walks the first half at the same rate as before, but the stream being with him, he rows $1\frac{1}{2}$ miles per hour more than in going, and accomplishes the whole distance in $3\frac{1}{2}$ hours. Find the rates of walking and rowing.

Ans. *$4\frac{1}{2}$ miles per hour walking; $4\frac{1}{2}$, rowing at first.*

116. Reduce the surds $\sqrt{300}$ and $\sqrt{75}$ to the same radical, and find their sum.

117. Find two numbers in the ratio of 5 to 6, such that their sum has to the difference of their squares the ratio of 1 to 7. *Ans. 35; 42.*

118. Solve the quadratic $\frac{x}{7}+\frac{21}{x+5}=\frac{23}{7}$.

119. An orange peddler bought a number of oranges at the rate of 5 for 2 cents. He then arranged the good and bad in two separate baskets, containing equal numbers, and sold the one basketful at 3 for 1 cent, and the other at 3 for 2 cents. In selling them, he was told he would make no profit upon them; but when he had sold the whole he found he had gained 6 cents. How many oranges did he buy and sell? *Ans. 360.*

120. Divide $4x-12$ by $\frac{1}{2}\sqrt{x-3}$. *Ans.* $8\sqrt{x-3}$.

121. A and B engage in partnership with a capital of \$5000; A has his money in for 3 months, and B for 2 months, and each at last realizes \$4950 of capital and profit. What was the original contribution of each? *Ans. A, \$2250; B, \$2750.*

122. The seventh term of a geometrical progression is 1000, and the common rate is $\frac{104}{100}$. What is the first term? *Ans. 790.31+*

123. A certain rectangle contains 300 square feet; a second rectangle is 8 feet shorter and 10 feet broader, and also contains 300 square feet; what is the length and breadth of the first rectangle? *Ans. 20 feet; 15 feet.*

124. What two numbers are such that their product is 45, and the difference of their squares is to the square of their difference as 7 is to 2? *Ans. 9 and 5.*

125. Two detachments of infantry being ordered to a station 39 miles from their present quarters, begin their march at the same time; but one party, by marching $\frac{1}{4}$ of a mile per hour faster than the other, arrive there an hour sooner. Required their rates of marching. *Ans.* $3\frac{1}{4}$ *and 3 miles per hour.*

APPENDIX.

SECTION XLVII.

GENERALIZATION.

364. A **General Problem** is one in which all the quantities are represented by letters.

The result obtained by the solution of a general problem expresses the value of the unknown in the terms of the known quantities.

365. **Generalization** is the process of solving a general problem, and interpreting the resulting expression.

The formulas, or general expressions, derived from the solution of general problems, when interpreted, form rules for the solution of all similar problems.

366. A problem is **generalized** when letters are made to represent its known quantities.

EXERCISES.

367.—Ex. 1. The sum of two numbers is s, and their difference is d. What are the two numbers?

SOLUTION. Let x = the greater number, and y = the less; then, by the conditions, $x+y=s$, and $x-y=d$.

Adding equations (2) and (1), and subtracting (2) from (1), we obtain equations (3) and (4). Whence, $x=\frac{s+d}{2}$ and $y=\frac{s-d}{2}$. But s and d may be any quantities whatever; hence, the values of x and y are general, and

x = *the greater*;
y = *the less.*

$$x+y=s \quad (1)$$
$$x-y=d \quad (2)$$
$$2x=s+d \quad (3)$$
$$2y=s-d \quad (4)$$
$$x=\frac{s+d}{2} \quad (5)$$
$$y=\frac{s-d}{2} \quad (6)$$

equations (5) and (6) are formulas for finding two numbers when their sum and difference are given. Hence, the following,—

368. Rule for finding two numbers from their sum and difference.—*Add the difference to the sum, and divide by 2, for the greater of the two numbers; subtract the difference from the sum, and divide by 2, for the less of the two numbers.*

1. The sum of two numbers is 391, and their difference is 53. What is the greater number?

2. A and B hire a pasture together for \$65, and A is to pay \$13 more than B. How much is each to pay?

Ans. A, \$39; B, \$26.

3. In an election, the aggregate of votes for A and B was 9637, and B's majority over A was 593. How many votes did each receive?

369.—Ex. 1. Divide the sum S among A, B and C, in the proportion of the numbers m, n and p.

SOLUTION. Let x = A's share; then $m:n::x:\frac{nx}{m}$, and $m:p::x:\frac{px}{m}$. Hence, $\frac{nx}{m}$ represents B's share, and $\frac{px}{m}$ represents C's share. By the conditions, $x+\frac{nx}{m}+\frac{px}{m}=S$; whence, $\frac{mS}{m+n+p}$, $\frac{nS}{m+n+p}$ and $\frac{pS}{m+n+p}$ are expressions for their several shares, and, being general, can be applied in the solution of all similar problems.

$$x = \textit{A's share};$$

$$m:n::x:\frac{nx}{m} = \textit{B's share};$$

$$m:p::x:\frac{px}{m} = \textit{C's share}.$$

$$x+\frac{nx}{m}+\frac{px}{m}=S$$

$$x=\frac{mS}{m+n+p}$$

$$\frac{nx}{m}=\frac{nS}{m+n+p}$$

$$\frac{px}{m}=\frac{pS}{m+n+p}$$

Hence, the following,—

370. Rule for dividing a sum into parts proportional to given numbers.—*Multiply the sum by each of the proportional numbers, and divide the several products by the sum of the proportional numbers.*

1. Divide \$5400 among A, B and C in proportion to the numbers 7, 9 and 11.

2. Three men are in company. A puts in \$8 as often as B puts in \$5, and as often as C puts in \$3. They gain \$1200; what is each man's share of it?

Ans. A's, \$600; B's, \$375; C's, \$225.

3. Four men, A, B, C and D, hire a pasture in common. A puts in 6 oxen for 5 months, B puts in 5 oxen for 3 months, C puts in 5 oxen for 5 months, and D puts in 5 oxen for 4 months. They are to pay \$90; how much is each man's share of it?

371.—Ex. 1. A cistern can be filled by three pipes; by the first in 2 hours, by the second in 5 hours, and by the third in 10 hours. In what time can it be filled by all the pipes running together? Generalize the problem.

x = *the number of hours in the time required;*

$\frac{1}{x}$ = *the part filled in 1 hour.*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$$

$$bcx+acx+abx=abc$$

$$(ab+ac+bc)x=abc$$

$$x=\frac{abc}{ab+ac+bc}$$

SOLUTION. Represent in the problem, 2, 5 and 10 by a, b and c respectively, and it becomes general.

Let x = the time required for all to fill the cistern; then $\frac{1}{x}$ = the part of the cistern filled in one hour by all running together, and, by the conditions, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$.

Clearing of fractions, factoring and dividing, we have the general answer, $\frac{abc}{ab+ac+bc}$ hours. That is,

The required time for three agencies operating together to accomplish a certain result, *is the product of the given times, divided by the sum of the products of the times taken two and two.*

Substituting for the letters, in the general answer, their respective values, we have the particular answer, $1\frac{1}{4}$ hours.

2. Two men are employed to do a piece of work; the first can do it in 10 days, and the second in 15 days. How many days would it take both, working together? Generalize the problem.

3. What principal at interest at 6 per cent. will, in 3 years, amount to $472? If a represent the amount, r the rate per cent. and t the time, what will be the general answer?

Particular Ans. $400; *general Ans.* $\frac{100a}{100+rt}$.

4. A gentleman distributing some money among some beggars, found that, in order to give them 8 cents apiece, he should want 5 cents more; he therefore gave them 7 cents each, and had 4 cents left. How many beggars were there? Generalize the problem.

Particular Ans. 9; *general Ans.* $\frac{b+d}{a-c}$.

5. The fore wheel of a carriage is 12 feet in circumference, and the hind wheel 18 feet; what will be the number of feet passed over, when the fore wheel has made 40 revolutions more than the hind wheel? If n denote the number of revolutions, a the circumference of the fore wheel, and b the circumference of the hind wheel, what will be the general answer?

Particular Ans. 1440; *general Ans.* $\frac{abn}{b-a}$.

6. A workman agreed to work 40 days. For each day he worked he was to receive $2, and for each day he was idle he was to forfeit $.50. He received $50. How many days did

he work? If n denote the given number of days, a the number of dollars for each day he worked, b the number of dollars forfeited for each day he was idle, and c the amount received at the end of the time, what will be the general answer?

Particular Ans. 28; *general Ans.* $\dfrac{c+bn}{a+b}$.

7. A merchant has two kinds of tea; the one cost 90 cents per pound, the other 50 cents. He wishes to mix both kinds together in such quantities that he may have 50 pounds, and each pound, without profit or loss, may be sold for 80 cents. How much must he take of each to make up the mixture? Generalize the problem.

Particular Ans. $37\frac{1}{2}$ *lbs. of the best;* $12\frac{1}{2}$ *lbs. of the other.*

General Ans. $\dfrac{(c-b)n}{a-b}$ *lbs. of the best;* $\dfrac{(a-c)n}{a-b}$ *of the other.*

SECTION XLVIII.

NEGATIVE SOLUTIONS.

372. Negative Solutions are such as produce results which have the minus sign.

EXERCISES.

373.—Ex. 1. The length of a garden is 8 rods and its width is 5 rods. How much must be added to the length, that the garden may contain 30 square rods?

SOLUTION. Let x = the quantity which must be added. Then, $(8+x)5$ = area of garden; hence, $40+5x=30$; whence, $5x=-10$, or $x=-2$, which satisfies the conditions of the problem algebraically, but not arithmetically.

x = *the quantity;*
$(8+x)5$ = *area of garden.*
$40+5x=30$
$5x=-10$
$x=-2$

But, since *adding* a negative quantity is equivalent to *subtracting* an

equal positive quantity, the negative result indicates that the problem, to be consistent arithmetically, should read:

The length of a garden is 8 rods, and its width is 5 rods; how much must be subtracted from its length that the garden may contain 30 square rods?

2. A father was 50 years old when his son was 20; in how many years afterwards was the father four times as old as the son?

SOLUTION. Let x = the number of years. Then, $20+x=\frac{50+x}{4}$; whence, $x=-10$. But the problem, in the exact sense of its enunciation, is impossible, for when the father was 50 years old and the son 20, the father was less than four times as old as the son; hence, the time when he was just four times as old as the son must have been *before* the given time.

$$x = \textit{the number.}$$
$$20+x=\frac{50+x}{4}$$
$$80+4x=50+x$$
$$3x=-30$$
$$x=-10$$

The -10 must then be understood as indicating that the time was *before* and not *after*, and the enunciation of the problem should be modified accordingly.

From these illustrations may be drawn the following inferences.

374.—1. *The negative solution of a problem by an equation of the first degree indicates some inconsistency or absurdity in the conditions of the problem as enunciated.*

2. *That in such cases an analogous problem consistent in its conditions can generally be formed by changing the terms of the absurd condition to those entirely opposite.*

Solve, interpret and properly modify the enunciation of the following problems:

1. A man when he was married was 45 years old, and his wife 20. How many years before was he twice as old as she?

Ans. -5.

The -5 indicates a wrong condition; that is, their ages bore the given relation 5 years *after*, not *before*, their marriage.

2. A had \$150 and B \$120. They each gained a certain sum, when it was found that A's money was to B's in the ratio of 3 to 2. What did each gain?

3. What number is that whose fourth part exceeds its third part by 16? *Ans.* -192.

That is, the enunciation contains a contradiction; it should read: What number is that whose third part exceeds its fourth part by 16?

4. What fraction is such that if 2 be added to its numerator its value is $\frac{1}{4}$, or if 2 be added to its denominator its value is $\frac{1}{2}$?

Ans. $\frac{-5}{-12}$.

5. A man worked 12 weeks, having his wife with him 7 weeks, and received \$46. He afterwards worked 8 weeks, having his wife with him 5 weeks, and received \$30. How much did he earn per week for himself, and how much did his wife earn? *Ans. Man, \$5; wife, −\$2.*

That is, the man earned \$5 per week for himself, and was charged \$2 per week for expense of his wife, when she was with him.

SECTION XLIX.

INDETERMINATE AND IMPOSSIBLE PROBLEMS.

375. Zero, or the symbol 0, may be used to denote the absence of value, or to represent a quantity less than any assignable value.

376. Infinity, or the symbol ∞, is used to denote a quantity greater than any assignable value.

A quantity which is less than any assignable value is said to be infinitely small.

377. An equation of the first degree with one unknown quantity may be reduced to the general form,

$$ax = b;$$

whence, $$x = \frac{b}{a};$$

in which we know that if a and b are both finite, the value of the unknown quantity will be a determinate finite quantity. But, if a or b, or both, be equal to 0, the values of the unknown quantity assume peculiar forms.

1. Let $b = 0$, then the value of x becomes

$$\frac{0}{a} = 0,$$

where a denotes some finite quantity.

For, with a given denominator, the less the numerator of a fraction, the less is the value of the fraction; and when the numerator becomes 0, the value of the fraction becomes infinitely small, or 0. Hence the solution is not possible in finite numbers.

2. Let $a = 0$, then the value of x becomes

$$\frac{a}{0} = \infty.$$

For, with a given numerator, the less the denominator of a fraction, the greater is the value of the fraction; and when the denominator becomes 0, the value of the fraction becomes infinitely large, or ∞. Hence, it is evident that in this case no finite value satisfies the equation.

3. Let $a = 0$ and $b = 0$, then the equation becomes

$$x = \frac{0}{0}.$$

This is equivalent to $0 \times x = 0$, an equation which any finite value whatever of x will satisfy. The solution is then indeterminate.

When, however, $\frac{0}{0}$ is derived from a fraction whose terms contain a common factor, the fraction may have a determinate value. Thus, let $x = \frac{a^2 - 1}{a^2 + a - 2}$, and $a = 1$; the value of x is then $\frac{0}{0}$. Dividing the terms of the fraction by $a - 1$, we have $x = \frac{a+1}{a+2} = \frac{2}{3}$. In this case the value of x is determinate.

378. An **Indeterminate Problem** is one whose conditions are satisfied by different values of the same unknown quantity.

Such a problem will be found to produce either an equation whose members are merely the repetition the one of the other, or a less number of independent equations than there are unknown quantities to be determined.

379. An **Impossible Problem** is one whose conditions are absurd or contradictory.

EXERCISES.

380.—Ex. 1. What number is that the sum of whose $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ is equal to the number increased by its $\frac{1}{12}$?

SOLUTION. Let x = the number; then, by the conditions, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=x+\frac{x}{12}$. Clearing of fractions, we have equation (1), and, by uniting, we have the identical equation (2). Transposing and factoring, we have (3) and (4); whence (5) and (6), or $x=\frac{0}{0}$. The problem is therefore indeterminate.

$$x = \textit{the number}.$$

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=x+\frac{x}{12}$$

$$6x+4x+3x=12x+x \qquad (1)$$

$$13x=13x \qquad (2)$$

$$13x-13x=0 \qquad (3)$$

$$(13-13)x=0 \qquad (4)$$

$$0\times x=0 \qquad (5)$$

$$x=\frac{0}{0} \qquad (6)$$

2. What number is such that the sum of its $\frac{1}{4}$ and its $\frac{2}{3}$, diminished by 1, is equal to its $\frac{11}{12}$ increased by 2?

SOLUTION. Let x = the number; then, by the conditions, $\frac{x}{4}+\frac{2x}{3}-1=\frac{11x}{12}+2$. Clearing of fractions, we have equation (1), and by uniting, transposing and factoring, we have equations (2) and (3). Whence (4) and (5), or $x=\infty$, which indicates that no finite number will satisfy the conditions. The problem is, therefore, impossible.

$$x = \textit{the number}.$$

$$\frac{x}{4}+\frac{2x}{3}-1=\frac{11x}{12}+2$$

$$3x+8x-12=11x+24 \qquad (1)$$

$$11x-11x=36 \qquad (2)$$

$$(11-11)x=36 \qquad (3)$$

$$0\times x=36 \qquad (4)$$

$$x=\frac{36}{0}=\infty \qquad (5)$$

3. Given $y+z=21$ and $y-x=13$, to find x and y.

SOLUTION. Subtracting the second equation from the first, we have $x+z=8$, an equation which admits of any number of values of x and z. The problem is therefore indeterminate.

4. Given $x+y=14$, $x-y=2$ and $\dfrac{3x}{y}=3$.

SOLUTION. From the first and second equations, we find $x=8$ and $y=6$. But the third equation requires 3 times x divided by y to be 3, which cannot be fulfilled for those values of x and y. The problem therefore is impossible.

5. A piratical vessel is 10 miles ahead of a sloop of war, and the two are sailing at equal rates per hour. In how many hours will the sloop overtake the pirate?

Ans. $x=\dfrac{10}{0}=\infty$. *Impossible.*

6. What number is such that if 3 be subtracted from twice the number, the remainder will be equal to one-third of the excess of 6 times the number over 9?

Ans. $x=\dfrac{0}{0}$. *Indeterminate.*

7. A man being asked how many fish he had caught, replied, "If 5 be added to one-third of the number that I caught yesterday, it will make half the number I have caught to-day; or if 5 be subtracted from three times this half, it will leave the number I caught yesterday." How many were caught each day?

Ans. Impossible.

8. I have money in two purses; that in the one is to that in the other in the ratio of 5 to 6; if $\frac{1}{6}$ of the money in the second were to be taken out, the money in the two purses would be equal. How many dollars are there in each?

Ans. Indeterminate

Test Questions.

381.—1. What is a *General Problem?* When is a problem generalized? What is generalization? How may the formulas derived by generalization be interpreted?

2. What are *Negative Solutions?* How are negative quantities derived? What does a negative solution of a problem by an equation of the first degree indicate? How can a problem leading to a negative result be generally changed to a problem consistent in its conditions?

3. What is an *Indeterminate Problem?* An impossible problem? What is zero used to denote? Infinity?

SECTION L.

IMAGINARY QUANTITIES.

382. An **Imaginary Quantity** is an indicated even root of a negative quantity.

Thus, $\sqrt{-a}$ and $\sqrt[4]{-16}$ are imaginary quantities, and symbolize processes which it is impossible to perform.

They are, however, of considerable use in mathematical analysis, and when subjected to certain rules of calculation, they lead to possible and valuable results.

383. The addition and subtraction of imaginary quantities are performed by the same rules that apply to other radicals; but with regard to their multiplication or division, the ordinary rules require some modification.

384. Principles.—1. *Every imaginary quantity may be resolved into two factors,—one a real quantity, and the other the imaginary expression* $\sqrt{-1}$, *or* $\sqrt[2n]{-1}$.

For every negative quantity may be regarded as the product of two factors, one of which is -1.

Thus, since $-a=a\times(-1)$, we have $\sqrt{-a}=\sqrt{a\times(-1)}=\sqrt{a}\times\sqrt{-1}$; also, since $-4=4\times(-1)$, we have $\sqrt{-4}=\sqrt{4\times(-1)}=\sqrt{4}\times\sqrt{-1}=2\sqrt{-1}$.

The real factor is called the **Coefficient** of the imaginary factor $\sqrt{-1}$.

2. *The product of two imaginary quantities is real, and the sign before the radical is the reverse of that obtained by the common rule.*

For the ambiguity in the signs to be prefixed to an even root by the common rule (Prin. 2, Art. 244) is removed when we know the factors which compose the quantity whose root is taken. These factors of an imaginary quantity we may know, as has already been shown. Thus,

$$\begin{aligned}\sqrt{-a}\times\sqrt{-b}&=(\sqrt{a}\times\sqrt{-1})(\sqrt{b}\times\sqrt{-1})\\&=\sqrt{ab}\times(-1)\\&=-\sqrt{ab}.\end{aligned}$$

In like manner it may be shown that

$$-\sqrt{-a}\times(-\sqrt{-b})=-\sqrt{ab};$$

$$\text{also,}\quad \sqrt{-a}\times(-\sqrt{-b})=+\sqrt{ab}.$$

3. *The quotient of one imaginary quantity divided by another is real, and has the sign before the radical the same as that obtained by the common rule.*

$$\text{Thus,}\quad \frac{\sqrt{-a}}{\sqrt{-b}}=\frac{\sqrt{a}\times\sqrt{-1}}{\sqrt{b}\times\sqrt{-1}}=\sqrt{\frac{a}{b}};$$

$$\text{also,}\quad \frac{-\sqrt{-a}}{\sqrt{-b}}=\frac{-\sqrt{a}\times\sqrt{-1}}{\sqrt{b}\times\sqrt{-1}}=-\sqrt{\frac{a}{b}}.$$

385. By attention to these principles, the algebraic processes with imaginary quantities may be readily performed.

PROBLEMS.

Ex. 1. Multiply $5\sqrt{-3}$ by $\sqrt{-2}$.

SOLUTION.

$$5\sqrt{-3}=5\sqrt{3}\times\sqrt{-1} \text{ and } \sqrt{-2}=\sqrt{2}\times\sqrt{-1}.$$

$$(5\sqrt{3}\times\sqrt{-1})(\sqrt{2}\times\sqrt{-1}=5\sqrt{6}\times(\sqrt{-1})^2=-5\sqrt{6}.$$

2. Multiply $3\sqrt{-2}$ by $2\sqrt{-1}$. *Ans.* $-6\sqrt{2}$.

3. Multiply $-7\sqrt{-a}$ by $-\sqrt{-ab}$. *Ans.* $-7a\sqrt{b}$.

4. Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$. *Ans.* $ab(\sqrt{-1})^2=-ab$.

5. Multiply $3-\sqrt{-2}$ by $3+\sqrt{-2}$. *Ans.* 11.

6. Multiply $1-\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* $-2\sqrt{-1}$.

7. Divide $\sqrt{-ab}$ by $\sqrt{-a}$.

SOLUTION.

$$\sqrt{-ab}=\sqrt{ab}\times\sqrt{-1} \text{ and } \sqrt{-a}=\sqrt{a}\times\sqrt{-1}; \quad \frac{\sqrt{ab}\times\sqrt{-1}}{\sqrt{a}\times\sqrt{-1}}=\sqrt{b}.$$

8. Divide $a\sqrt{-1}$ by $b\sqrt{-1}$. *Ans.* $\frac{a}{b}$.

9. Divide $10\sqrt{-14}$ by $2\sqrt{-7}$. *Ans.* $5\sqrt{2}$.

10. Divide $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* $\sqrt{-1}$.

11. Divide $-8\sqrt{-4}$ by $-2\sqrt{-3}$. *Ans.* $4\sqrt{\frac{4}{3}}$.

12. Multiply $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$. *Ans.* a^2+b^2.

SECTION LI.

BINOMIAL THEOREM.

386. The **Binomial Theorem** expresses the simplest method known of writing out the different powers of any binomial. This method, discovered by Sir Isaac Newton, may be deduced from the first few powers of any binomials, as $a+b$ and $a-b$.

1. Let $a+b$ be raised to the 2d, 3d, 4th and 5th powers, by actual multiplication.

SOLUTION.

$$
\begin{array}{ll}
\textit{1st power,} & a+b \\
 & \underline{a+b} \\
 & a^2+ab \\
 & \qquad ab+b^2 \\
\hline
\textit{2d power,} & a^2+2ab+b^2 \\
 & \underline{a+b} \\
 & a^3+2a^2b+ab^2 \\
 & \qquad a^2b+2ab^2+b^3 \\
\hline
\textit{3d power,} & a^3+3a^2b+3ab^2+b^3 \\
 & \underline{a+b} \\
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \qquad a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
\textit{4th power,} & a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
 & \underline{a+b} \\
 & a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
 & \qquad a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\
\hline
\textit{5th power,} & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{array}
$$

2. Let $a-b$ be raised to the 2d, 3d, 4th and 5th powers, by actual multiplication.

SOLUTION.

$$\begin{array}{ll}
\textit{1st power,} & a-b \\
 & \underline{a-b} \\
 & a^2-ab \\
 & \underline{\quad -ab+b^2} \\
\textit{2d power,} & a^2-2ab+b^2 \\
 & \underline{a-b} \\
 & a^3-2a^2b+ab^2 \\
 & \underline{\quad -a^2b+2ab^2-b^3} \\
\textit{3d power,} & a^3-3a^2b+3ab^2-b^3 \\
 & \underline{a-b} \\
 & a^4-3a^3b+3a^2b^2-ab^3 \\
 & \underline{\quad -a^3b+3a^2b^2-3ab^3+b^4} \\
\textit{4th power,} & a^4-4a^3b+6a^2b^2-4ab^3+b^4 \\
 & \underline{a-b} \\
 & a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\
 & \underline{\quad -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5} \\
\textit{5th power,} & a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5
\end{array}$$

387. These different powers of $a+b$ and $a-b$ show that certain invariable laws govern the expansion of a binomial.

1. *The first, or leading, term of the binomial appears in every term of the power, except the last. Its exponent in the first term is the same as the index of the power, and in the following terms it decreases regularly by one.*

2. *The second term of the binomial appears in every term of the power, except the first. Its exponent in the second term is*

one, and in the following terms it increases regularly by one, becoming in the last term equal to the index of the power.

Thus, in the fifth power of each of the given binomials,

The exponents of a are	5,	4,	3,	2,	1.	
The exponents of b are		1,	2,	3,	4,	5.

3. *The coefficient of the first term is one; that of the second is the same as the index of the power; and, in general, the coefficient of any term, multiplied by the exponent of the leading quantity of that term, and divided by the exponent of the following quantity increased by one, equals the coefficient of the next term.*

Thus, in the fifth power of $a+b$, the coefficient of the first term is 1; that of the second term is the same as the exponent of the power, or 5; the coefficient of the second term, 5, multiplied by 4, the exponent of a, the leading quantity of that term, and divided by 2, the exponent of b increased by one $\left(=\frac{5\times 4}{2}=10\right)$, is the coefficient of the third term; and, in like manner, 10 is the coefficient of the fourth term; 5 is the coefficient of the fifth term; and 1 is the coefficient of the last term.

4. *When both terms of the binomial are positive, all the terms of the power are positive; but when the second term is negative, all the odd terms, counted from the left, are positive, and all the even terms negative.*

It will be seen that the coefficients are repeated in the inverse order after passing the middle term or terms, so that most of the coefficients can be written without calculation.

When the number of terms is even, the two middle terms have the same coefficient.

It may also be observed that the number of terms is always one more than the exponent of the power, and tbat the sum of the exponents in any term is equal to the exponent of the power.

388. Combining the principles given in the preceding article, we have, when the exponent of the power is n, the general development,

$$(a+b)^n =$$
$$a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\times 3}a^{n-3}b^3 + \ldots. nab^{n-1} + b^n,$$

which is the Binomial Formula, and can be used in the expansion of any binomial.

PROBLEMS.

1. Raise $a+x$ to the fourth power.

SOLUTION.

Letters and exponents,	a^4,	a^3x,	a^2x^2,	ax^3,	x^4
Coefficients,	1	$+4$	$+6$	$+4$	$+1$
Combining,	a^4	$+4a^3x$	$+6a^2x^2$	$+4ax^3$	$+x^4$

2. Raise $x-y$ to the fourth power.
Ans. $x^4-4x^3y+6x^2y^2-4xy^3+y^4$.

3. Raise $x+y$ to the sixth power.

4. What is the eighth power of $x-y$?

5. Find the fourth power of $1+x$.

SOLUTION.

Expanding the terms,	1^4	1^3x	1^2x^2	$1x^3$	x^4
Coefficients,	1	$+4$	$+6$	$+4$	$+1$
Rejecting the factor 1, *and combining,*	1	$+4x$	$+6x^2$	$+4x^3$	$+x^4$

6. Find the sixth power of $1-x$.
Ans. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.

7. Develop $(a-1)^5$. *Ans.* $a^5-5a^4+10a^3-10a^2+5a-1$.

389. The **Binomial Formula** also applies when either or both terms of a binomial have coefficients or exponents.

21

1. What is the third power of $2a-3b$?

SOLUTION.

Expanding terms, $(2a)^3-3(2a)^2(3b)+3(2a)(3b)^2-(3b)^3$

Multiplying factors, $8a^3-36a^2b+54ab^2-27b^3$

2. Develop $(a^2+b^3)^4$. *Ans.* $a^8+4a^6b^3+6a^4b^6+4a^2b^9+b^{12}$.

3. Develop $(1+2x)^3$. *Ans.* $1+6x+12x^2+8x^3$.

4. Expand $(2a-1)^4$. *Ans.* $16a^4-32a^3+24a^2-8a+1$.

5. What is the square of $9x+\frac{1}{x}$? *Ans.* $81x^2+18+\frac{1}{x^2}$.

6. What is the cube of $2ax-3bc^2$?
Ans. $8a^3x^3-36a^2b^2cx^2+54ab^2c^4x-27b^3c^6$.

390. A **Polynomial** may be raised to any power by putting it under the binomial form, and then applying the general formula.

1. Find the cube of $a+b+c$.

SOLUTION.

Putting $a+b+c$ *under the binomial form, we have*

$$a+(b+c).$$

Expanding $[a+(b+c)]^3$, *we have*

$$a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3.$$

Expanding and multiplying factors, we have

$$a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3.$$

2. Find the square of $a-b+y$.
Ans. $a^2-2ab+2ay+b^2-2by+y^2$.

3. Find the square of $a-b-c$.
Ans. $a^2-2ab+b^2-2ac+2bc+c^2$.

SECTION LII.

LOGARITHMS.

391. The **Logarithm** of a number is the exponent of the power to which a constant number must be involved to produce the given number.

Thus, if 8 is the constant number, 2 is the logarithm of 64, because $8^2=64$; 3 is the logarithm of 512, because $8^3=512$.

392. The **Base** of logarithms is the constant number which must be involved to produce the numbers.

393. Common Logarithms are those whose base is 10. Hence, in the common system,

0 is the logarithm of 1, since 10^0 equals 1;
1 is the logarithm of 10, since 10^1 equals 10;
2 is the logarithm of 100, since 10^2 equals 100;
3 is the logarithm of 1000, since 10^3 equals 1000;

and, generally, if n be any positive integer, and *log.* stand for logarithm, we have

$$log.\ 10^n=n.$$

Again, by means of negative exponents,

-1 is the logarithm of .1, since 10^{-1} equals .1;
-2 is the logarithm of .01, since 10^{-2} equals .01;
-3 is the logarithm of .001, since 10^{-3} equals .001;

and, generally, if n be any negative integer, we have

$$log.\ \frac{1}{10^n}=-n.$$

It thus appears that the logarithm of any number between 0 and 10 must be a fraction; between 10 and 100, 1 plus a fraction; between 100 and 1000, 2 plus a fraction, and so on.

The logarithms of numbers which are not exact powers can only be obtained approximately, and are usually expressed by decimals.

394. The **Characteristic** of a logarithm is the integral part of its expression.

The decimal part is sometimes called the **mantissa.**

395. Principles.—1. *The characteristic of the logarithm of any number greater than* 1, *is one less than the number of integral orders in the given number.*

For, the logarithm of 1 is 0, of 10 is 1, of 100 is 2, of 1000 is 3, and so on.

2. *The characteristic of the logarithm of any number less than* 1, *expressed as a decimal, is one more than the number of ciphers between the decimal point and the first numeral figure in the decimal.*

For, the logarithm of .1 is -1; of .01 is -2; of .001 is -3, and so on.

When the characteristic of the logarithm is a negative number, it is distinguished by being written with a short horizontal line over it.

Thus, $\bar{1}$ is written instead of -1, $\bar{2}$ instead of -2, etc.

3. *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of those numbers.*

For, let m and n be any two numbers, x and y their respective logarithms, and a the base of the system. Then, by the definition of a logarithm, we have

$$a^x = m, \quad a^y = n.$$

Multiplying these equations the one by the other, member by member, we have

$$a^{x+y} = mn,$$

in which $x+y$ is the logarithm of the product mn.

4. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend diminished by that of the divisor.*

For, dividing the equation $a^x = m$ by the equation $a^y = n$, member by member, we have

$$a^{x-y} = \frac{m}{n}$$

in which $x-y$ is the logarithm of the quotient $\frac{m}{n}$.

5. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For, raising both members of the equation $a^x = m$ to any power p, we have

$$a^{xp} = m^p,$$

in which xp is the logarithm of m raised to the power p.

6. *The logarithm of the root of any number is equal to the logarithm of that number divided by the index of the root.*

For, extracting the rth root of both members of the equation $a^x = m$ we have

$$a^{\frac{x}{r}} = \sqrt[r]{m}$$

in which $\frac{x}{r}$ is the logarithm of $\sqrt[r]{m}$.

7. *Numbers integral, decimal or mixed, having the same succession of numeral figures, have logarithms with the same decimal part.*

For, the decimal part of a logarithm of a number is the same as that of the product or the quotient of the number, multiplied or divided by 10, 100, 1000, etc.; and since the logarithms of 10, 100, 1000, etc., are 1, 2, 3, etc., the effect of the multiplication or division must be to change the characteristic of the logarithm of the number, and not to change the decimal part.

TABLES OF LOGARITHMS.

396. A **Table** of logarithms contains the computed logarithms of all integers between 1 and some given number.

The computation of the logarithms is generally made by means of a series developed by the aid of the binomial theorem, and is too abstruse for a work of this kind.

397. The **Table** given on the next two pages gives the decimal part of the common logarithms of the series of natural numbers from 10 to 999, carried to five decimal orders. The characteristics are not given in the table, but are left to be supplied by inspection. (Prin. 1 and 2, Art. 395).

Table of Common Logarithms.

No.	0	1	2	3	4	5	6	7	8	9	D.
10	00000	0432	0860	1284	1703	2119	2531	2938	3342	3743	414
11	04139	4532	4922	5308	5690	6070	6446	6819	7188	7555	378
12	07918	8279	8636	8991	9342	9691	0037	0380	0721	1059	348
13	11394	1727	2057	2385	2710	3033	3354	3672	3988	4301	322
14	14613	4922	5229	5534	5836	6137	6435	6732	7026	7319	300
15	17609	7898	8184	8469	8752	9033	9312	9590	9866	2140	280
16	20412	0683	0952	1219	1484	1748	2011	2272	2531	0789	263
17	23045	3300	3553	3805	4055	4304	4551	4797	5042	5285	248
18	25527	5768	6007	6245	6482	6717	6951	7181	7416	7646	235
19	27875	8103	8330	8556	8780	9003	9226	9447	9667	9885	223
20	30103	0320	0535	0750	0963	1175	1387	1597	1806	2015	212
21	32222	2428	2634	2838	3041	3244	3445	3646	3846	4044	202
22	34242	4439	4635	4830	5025	5218	5411	5603	5793	5984	193
23	36173	6361	6549	6736	6922	7107	7291	7475	7658	7840	185
24	38021	8202	8382	8561	8739	8917	9094	9270	9445	9620	177
25	39794	9967	0140	0312	0483	0654	0824	0993	1163	1330	170
26	41497	1664	1830	1996	2160	2325	2488	2651	2813	2975	164
27	43136	3297	3457	3616	3775	3933	4091	4248	4404	4560	158
28	44716	4871	5025	5179	5332	5484	5637	5788	5939	6090	152
29	46240	6389	6538	6687	6835	6982	7129	7276	7422	7567	147
30	47712	7857	8001	8144	8287	8430	8572	8714	8855	8996	142
31	49136	9276	9415	9554	9693	9831	9969	0106	0243	0379	138
32	50515	0651	0786	0920	1055	1188	1322	1455	1587	1720	134
33	51851	1983	2114	2244	2375	2504	2634	2763	2892	3020	130
34	53148	3275	3403	3529	3656	3782	3908	4033	4158	4283	126
35	54407	4531	4654	4777	4900	5023	5145	5267	5388	5509	122
36	55630	5751	5871	5991	6110	6229	6348	6467	6585	6703	119
37	56820	6937	7054	7177	7287	7403	7519	7634	7749	7864	116
38	57978	8092	8206	8320	8433	8546	8657	8771	8883	8995	113
39	59106	9218	9329	9439	5550	9660	9770	9879	9988	0097	110
40	60206	0314	0423	0531	0638	0746	0853	0959	1066	1172	107
41	61278	1384	1490	1595	1700	1805	1909	2014	2118	2221	105
42	62325	2428	2531	2634	2737	2839	2941	3043	3144	3246	102
43	63347	3448	3548	3649	3749	3849	3949	4048	4147	4246	100
44	64345	4444	4542	4640	4738	4836	4933	5031	5128	5225	98
45	65321	5418	5514	5610	5706	5801	5896	5992	6087	6181	95
46	66276	6370	6464	6558	6652	6745	6839	6932	7025	7117	93
47	67210	7302	7394	7486	7578	7669	7761	7852	7943	8034	91
48	68124	8215	8305	8395	8485	8574	8664	8753	8842	8931	90
49	69020	9108	9197	9285	9373	9461	9546	9636	9723	9810	88
50	69897	9984	0070	0157	0243	0329	0415	0501	0586	0672	86
51	70757	0842	0927	1012	1096	1181	1265	1349	1433	1517	84
52	71600	1684	1767	1850	1933	2016	2099	2181	2263	2346	83
53	72428	2509	2591	2673	2754	2835	2916	2997	3078	3159	81
54	73239	3320	3400	3480	3560	3640	3719	3799	3878	3957	80

Table of Common Logarithms.

No.	0	1	2	3	4	5	6	7	8	9	D.
55	74036	4115	4194	4273	4351	4429	4507	4586	4663	4741	78
56	74819	4896	4974	5051	5128	5205	5282	5358	5435	5511	77
57	75587	5664	5740	5815	5891	5967	6042	6118	6193	6268	76
58	76343	6418	6492	6567	6641	6716	6790	6864	6938	7012	74
59	77085	7159	7232	7305	7379	7452	7525	7591	7670	7743	73
60	77815	7887	7960	8032	8104	8176	8247	8319	8390	8462	72
61	78533	8604	8675	8746	8817	8888	8958	9029	9099	9169	71
62	79239	9309	9379	9449	9518	9588	9657	9727	9796	9865	69
63	79934	0003	0072	0140	0209	0277	0346	0414	0482	0550	68
64	80618	0686	0754	0821	0889	0956	1023	1090	1158	1224	67
65	81291	1358	1425	1491	1558	1624	1690	1757	1823	1889	66
66	81954	2020	2086	2151	2217	2282	2347	2413	2478	2543	65
67	82607	2672	2737	2802	2866	2930	2995	3059	3123	3187	64
68	83251	3315	3378	3442	3506	3569	3632	3696	3759	3822	63
69	83885	3948	4011	4073	4136	4198	4261	4323	4386	4448	62
70	84510	4572	4634	4696	4757	4819	4880	4942	5003	5065	62
71	85126	5187	5248	5309	5370	5431	5491	5552	5612	5673	61
72	85733	5794	5854	5914	5974	6034	6094	6153	6213	6273	60
73	86332	6392	6451	6510	6570	6629	6688	6747	6806	6864	59
74	86923	6982	7040	7099	7157	7216	7274	7332	7390	7448	58
75	87506	7564	7622	7679	7737	7795	7852	7910	7967	8024	58
76	88081	8138	8195	8252	8309	8366	8423	8480	8536	8593	57
77	88649	8705	8762	8818	8874	8930	8986	9042	9098	9154	56
78	89209	9265	9321	9376	9432	9487	9542	9597	9653	9708	55
79	89763	9818	9873	9927	9982	0037	0091	0146	0200	0255	55
80	90309	0363	0417	0472	0526	0580	0634	0687	0741	0795	54
81	90849	0902	0956	1009	1062	1116	1169	1222	1275	1328	53
82	91381	1434	1487	1540	1593	1645	1698	1751	1803	1855	53
83	91908	1960	2012	2065	2117	2169	2221	2273	2324	2376	52
84	92428	2480	2531	2583	2634	2686	2737	2788	2840	2891	51
85	92942	2993	3044	3095	3146	3197	3247	3298	3349	3399	51
86	93450	3500	3551	3601	3651	3702	3752	3802	3852	3902	50
87	93952	4002	4052	4101	4151	4201	4250	4300	4349	4399	50
88	94448	4498	4547	4596	4645	4694	4743	4792	4841	4890	49
89	94939	4988	5036	5085	5134	5182	5231	5279	5328	5376	49
90	95424	5472	5521	5569	5617	5665	5713	5761	5809	5856	48
91	95904	5952	5999	6047	6095	6142	6190	6237	6284	6332	47
92	96379	6426	6473	6520	6567	6614	6661	6708	6755	6802	47
93	96848	6895	6942	6988	7035	7081	7128	7174	7220	7267	46
94	97313	7359	7405	7451	7497	7543	7589	7635	7681	7727	46
95	97772	7818	7864	7909	7955	8000	8046	8091	8137	8182	45
96	98227	8272	8318	8363	8408	8453	8498	8543	8588	8632	45
97	98677	8722	8767	8811	8856	8900	8945	8989	9074	9078	45
98	99123	9167	9211	9255	9300	9346	9388	9432	9476	9520	44
99	99564	9607	9651	9695	9799	9782	9826	9870	9913	9957	44

398. The **Numbers** expressed by not more than two orders of figures are given in the column marked No., and opposite each number in the adjacent column is its corresponding logarithm.

The first two orders on the left of a number expressed by three orders of figures are given in the column marked No., and the third order is given at the top of another column on the same page.

The first order on the left of the mantissa is printed only once, and in the first column of logarithms, for all the ten logarithms in the horizontal line; and where a figure of the mantissa is printed in light-face type, the first order on the left of the mantissa is to be found in the 0 column in the line next below.

399. The **Average Difference** of the ten logarithms in the same horizontal line is given in the column marked D.

400. The mantissas of the logarithms of 1, 2, 3, etc., are the same as those of 10, 20, 30, etc. Hence, the preceding table is sufficient for the first 1000 natural numbers.

EXERCISES.

401.—Ex. 1. Find the logarithm of 131.

SOLUTION. The decimal part of the logarithm, from the table, corresponding to the given number, is

$$.11727$$

Prefixing the characteristic (Prin. 1, Art. 395), we have

$$2.11727 = \text{Log. of } 131.$$

2. Find the logarithm of 144. *Ans.* 2.15836

3. Find the logarithm of 99.

4. Find the logarithm of .0102

SOLUTION. The decimal part of the logarithm in the table, corresponding to the given number, is

$$.00860$$

Prefixing the characteristic (Prin. 2, Art. 375), we have

$$\bar{2}.00860$$

5. Find the logarithm of .0017 *Ans.* $\bar{3}.23045$

6. Find the logarithm of 1.29 *Ans.* 0.11059

7. Find the logarithm of 10.2731

SOLUTION. The logarithm corresponding to 10.2 is

$$1.00860$$

The difference in column D, on the same horizontal line as the decimal part of the logarithm, is 414, which being multiplied by 731, the remaining orders of the given number, we have

$$414 \times 731 = 302634.$$

Rejecting from the right of this product as many orders as there are figures in the multiplier 731, we have the part, 303, approximately obtained, to be added to the logarithm found. Adding, we have,

$$\begin{array}{r} 1.00860 \\ 303 \\ \hline 1.01163 \end{array} = \textit{Log. of } 10.2731$$

8. Find the logarithm of 1495. *Ans.* 3.1746

9. Find the logarithm of 10210.

10. Find the logarithm of .00763 *Ans.* $\bar{3}.88275$

11. Find the logarithm of .018414 *Ans.* $\bar{2}.26515$

402.—Ex. 1. Find the number whose logarithm is 2.98182

SOLUTION. The number in the table corresponding to the decimal part of the given logarithm is 959, which according to the characteristic must contain three integral orders. Hence, the required number is the integer 959.

2. Find the number whose logarithm is 1.61066

3. Find the number whose logarithm is 2.15836 *Ans.* 144.

4. Find the number whose logarithm is $\bar{3}.23045$ *Ans.* .0017

5. Find the number whose logarithm is 1.52634

6. Find the number whose logarithm is 2.57287 *Ans.* 374.

7. Find the number whose logarithm is 1.011702

SOLUTION. The number in the table whose logarithm is next less than the given logarithm is 10.2

The given logarithm,	*1.01170*	
The logarithm next less,	*1.00860; corresponding number, 10.2*	
Difference of logarithms,	$\frac{310}{414} =$	*747*
Difference from column D,		
Number required,		*10.2747*

8. Find the number whose logarithm is 1.60552

Ans. 40.32

9. Find the number whose logarithm is $\bar{3}.88275$

Ans. .007634

10. Find the number whose logarithm is 3.17464

11. Find the number whose logarithm is $\bar{1}.082426$

Ans. .1209

Multiplication by Logarithms.

403.—Ex. 1. Multiply 76.4 by 5.4

SOLUTION.

Logarithm of	*76.4*	= *1.88309*
Logarithm of	*5.4*	= *0.73239*
Logarithm of	*412.56*	= *2.61548*

404. Rule for Multiplying one Number by another by Logarithms.—*Add the logarithms of the factors, and find the number corresponding to the sum.*—(Prin. 3, Art. 395.)

PROBLEMS.

1. Multiply 5.3 by 2.8 *Ans. 14.84*
2. Multiply 3.26 by .0025 *Ans. .00815*
3. Multiply .25 by .003 *Ans. .00075*
4. Multiply 134 by 25.6 *Ans. 3430 4*
5. Multiply 1853 by 46.
6. Multiply .0051 by 2.3 *Ans. .01173*

Division by Logarithms.

405.—Ex. 1. Divide 59.45 by .0315

SOLUTION.

$$\begin{aligned} &\textit{Logarithm of } 59.45 = 1.77415 \\ &\textit{Logarithm of } .0315 = \underline{\bar{2}.49831} \\ &\textit{Logarithm of } 1887.1 = 3.27584 \end{aligned}$$

406. Rule for Dividing one Number by another by Logarithms.—*Subtract the logarithm of the divisor from the logarithm of the dividend, and find the number corresponding to the difference.*—(Art. 375.)

PROBLEMS.

1. Divide 875 by 25. *Ans.* 35.

2. Divide 410.4 by 5.4 *Ans.* 76.

3. Divide .008215 by .031 *Ans.* .265

4. Divide .0023808 by 3.72 *Ans.* .00064

5. Find the value of $\frac{3}{8}$ in a decimal expression.

SOLUTION.

The fraction $\frac{3}{8}$ is equal to 3 divided by 8.

$$\begin{aligned} &\textit{Logarithm of } 3 = 0.47712 \\ &\textit{Logarithm of } 8 = \underline{0.90309} \\ &\textit{Logarithm of } \tfrac{3}{8}, \textit{ or } .375 = \bar{1}.57403 \end{aligned}$$

6. Find the value of $3\frac{1}{4}$ or $\frac{13}{4}$ in a decimal expression. *Ans.* 3.25

7. Find the value of $\frac{15}{34}$ in a decimal expression of two orders. *Ans.* .44

8. Find the value of $\frac{12}{125}$ in a decimal expression. *Ans.* .096

Involution by Logarithms.

407.—Ex. 1. Find the second power of 6.4

SOLUTION.

$$\begin{array}{lrl} \textit{Logarithm of} & 6.4 & = 0.80618 \\ & & \underline{\qquad\quad 2} \\ \textit{Logarithm of} & 40.96 & = 1.61236 \end{array}$$

408. Rule for Involution by Logarithms.—*Multiply the logarithm of the number by the exponent of the power, and find the number corresponding to the product.*—(Art. 402.)

PROBLEMS.

1. Find the fourth power of 1.05 to four orders of decimals. *Ans. 1.2155*

2. Find the fifth power of 1.06 to three orders of decimals. *Ans. 1.338*

3. Find the cube of .25 *Ans. .015625*

4. Find the fifth power of .836 *Ans. .4083+*

5. Find the cube of 13. *Ans. 2197.*

6. Find the fifth power of 2.846 *Ans. 186.65*

Evolution by Logarithms.

409.—Ex. 1. Find the square root of 40.96

SOLUTION.

$$\begin{array}{lrl} \textit{Logarithm of} & 40.96 & = 1.61236 \\ \textit{Dividing by 2, we have} & & = 0.80618 \end{array}$$

whose corresponding number is 6.4

410. Rule for Evolution by Logarithms.—*Divide the logarithm of the number by the integer denoting the degree of the root, and find the number corresponding to the quotient.*—(Art. 402.)

PROBLEMS.

1. Find the square root of 1849. *Ans.* *43.*

2. Find the cube root of 2197. *Ans.* *13.*

3. Find the cube root of 10648. *Ans.* *22.*

4. Find the tenth root of 1024. *Ans.* *2.*

5. Find the fifth root .00009 to four orders of decimals.

6. Find the square root of .001849.

SOLUTION.

$$\textit{Logarithm of } .001849 = \bar{3}.26694$$

Adding -1 to $\bar{3}$, to make it divisible by the index of the root, and adding $+1$ to mantissa, to the offset, we have

$$\bar{4} + 1.26694,$$

and dividing by 2, we obtain

$$\bar{2}.63347,$$

whose corresponding number is .043.

That is, when the characteristic of the logarithm is negative, and not divisible by the index of the root, the characteristic may be increased by any number which will make it exactly divisible, provided we prefix an equal positive number to the mantissa of the logarithm.

7. Find the cube root of .015625 *Ans.* *.25*

8. Find the seventh root of .005846 *Ans.* *.4797*

9. Find the sixth root of .0432 to three orders of decimals. *Ans.* *.592*

10. What is the value of $\sqrt[5]{.034}$ to three orders of decimals? *Ans.* *.508*

11. What is the value of $\sqrt{.0000169}$ to four orders of decimals? *Ans.* *.0041*

12. Find the tenth root of 581.4 to two orders of decimals. *Ans.* *1.89*

COMPOUND INTEREST.

411. Compound Interest is interest reckoned on interest and principal combined, at specified intervals.

The intervals may be years, half years, quarters, etc., according as the interest is made part of the principal annually, semiannually, quarterly, etc.

412. Let p denote any principal at compound interest;

r, the rate of interest plus 1, or the ratio of increase by compound interest;

n, the number of years, or other intervals, for which interest is taken; and

M, the amount at the end of n years; and we have, in general terms,

$$M = pr^n.$$

Hence, $$p = \frac{M}{r^n},$$

and, $$r = \left(\frac{M}{p}\right)^{\frac{1}{n}}$$

413. The compound interest gained in n intervals is

$$M - p, \text{ or } pr^n - p = p\,(r^n - 1).$$

414. The **Computations** of compound interest are most readily performed by the use of logarithms.

Taking the logarithms of both members of the equation $M = pr^n$, we have the useful formulas:—

$$\log.\ M = \log.\ p + n \log.\ r \qquad (1)$$

$$\log.\ p = \log.\ M - n \log.\ r \qquad (2)$$

$$\log.\ r = \frac{\log.\ M - \log.\ p}{n} \qquad (3)$$

$$n = \frac{\log.\ M - \log.\ p}{\log.\ r} \qquad (4)$$

EXERCISES.

415. To how much will \$100 amount in 7 years, at 5% compound interest?

SOLUTION.

$$M = pr^n = \$100 \times 1.05^7$$

$$Log.\ 105 = 0.02119$$
$$7$$
$$Log.\ 105^7 = 0.14833$$
$$Log.\ 100 = 2.00000$$
$$Log.\ 140.73 = 2.14833$$

2. What will be the amount of \$1, at 7% compound interest, in 15 years? *Ans.* \$2.75

3. How much will be the amount of \$10, at 6% compound interest, in 7 years 6 months, the interest payable semi-annually?

4. How much will be the compound interest of \$500 for 5 years, at 6%? *Ans.* \$169.10

5. What sum of money at 6% compound interest will amount to \$150 in 10 years?

SOLUTION.

$$p = \frac{M}{r^n} = \frac{\$150}{1.06^{10}}$$

$$Log.\ 150 = 2.17609$$
$$Log.\ 1.06^{10} = .25310$$
$$Log.\ 83.75 = 1.92299$$

Ans. \$83.75

6. What sum of money at 6% compound interest will amount to \$1000 in 12 years?

7. What sum of money at 5% compound interest will amount to \$5000 in 50 years? *Ans.* \$435.98

8. What sum of money at 7% compound interest will amount to \$276 dollars in 15 years? *Ans. \$100.*

9. At what rate must \$100 be at compound interest to become \$276 in 15 years?

SOLUTION.

$$r=\left(\frac{M}{p}\right)^{\frac{1}{n}}=\sqrt[15]{\frac{276}{100}}$$

$$\begin{array}{rl} Log.\ 276 = & 2.44091 \\ Log.\ 100 = & \underline{2.00000} \\ 15)\ & \underline{.44091} \\ Log.\ 1.07\quad & .02939 \end{array}$$

$1.07-1=.07$, or 7%. *Ans.* 7%.

10. At what rate must \$10 be at compound interest to amount to \$14.80 in 10 years? *Ans. 4%.*

11. At what rate must \$50 be at compound interest to amount to \$137.50 in 15 years? *Ans. 7%.*

12. In how many years will \$100 amount to \$350, at 6% compound interest?

SOLUTION.

$$n=\frac{log.\ M-log.\ p}{log.\ r}$$

$$\begin{array}{ll} Log.\ 350 & = 2.54407 \\ Log.\ 100 & = \underline{2.00000} \\ Log.\ 1.06=0.02531 &)\underline{0.54407} \\ & \quad 21\frac{1256}{2531} \end{array}$$

Ans. $21\frac{1}{2}$ *years, nearly.*

13. In what time will \$75 become \$140, at 5% compound interest?

14. In what time will \$100 become \$200, or double itself, at 6% compound interest? *Ans. 11.89 years.*

15. In how many years will a sum of money treble itself, at $3\frac{1}{2}\%$ compound interest? *Ans. 32 years, nearly.*

416. Increase of Population in a country furnishes problems similar to those of compound interest.

Ex. 1. One person out of 46 is said to die every year in England, and one to every 33 is born. Without emigration, in how many years would the population double itself?

SOLUTION. Let 46×33, or 1518, persons be living at the beginning of any year. According to conditions, 46 persons will be born during the year, and 33 will die; hence, 1518 at the beginning of the year becomes, at the end of the year, 1531; that is, the population increases every year by the multiplier $\frac{1531}{1518}$; hence, in n years it increases by the multiplier $\left(\frac{1531}{1518}\right)^n$. Wherefore, if in n years the population is doubled,

$$\left(\frac{1531}{1518}\right)^n = 2; \text{ whence, } n = \frac{\log. 2}{\log. 1531 - \log. 1518} = 81.3$$

Hence, the population will be doubled in about $81\frac{1}{3}$ years.

2. At about what annual rate of increase will a country double its population in 29 years? *Ans.* $2\frac{4}{10}\%$.

3. The population of the United States in 1790 was about 3,900,000, and in 1870 about 39,000,000. What was the average increase for each decennial period?

Annuities.

417. An **Annuity** is a sum of money stipulated to be paid annually.

418. *To find the amount of an annuity for any number of years at compound interest.*

Let a denote the annuity;
r, the rate of interest plus 1, or the amount of a in one year;
n, the number of years a draws interest; and
A, the amount of a for n years.

The annuity becomes due at the end of each year. Thus, in $n-1$ years it will amount to ar^{n-1} (Art. 355), in $n-2$ years it will amount to ar^{n-2}; in $n-3$ years to ar^{n-3}, and so on, to the last annuity, which will be simply a.

Hence, the amount due at the end of n years is

$$a+ar+ar^2+ar^3+\ldots+ar^{n-1},$$

a geometrical progression whose sum, by Art. 356, gives

$$A=\frac{ar^n-a}{r-1}.$$

419. *To find the present worth of an annuity to continue for a certain number of years at compound interest.*

Let P denote the present worth; then the amount of P in n years must be equal to the amount of the annuity in n years; that is,

$$Pr^n=\frac{ar^n-a}{r-1};$$

whence,

$$P=\frac{ar^n-a}{r^n(r-1)}=\frac{a}{r-1}\left(1-\frac{1}{r^n}\right).$$

EXERCISES.

420.—Ex. 1. An annuity of \$50 a year has remained unpaid 7 years. What amount is due, allowing compound interest at 4%? *Ans.* \$395.

2. What amount is due on a yearly pension of \$100 which has remained unpaid 6 years, at 5% compound interest? *Ans.* \$680.20

3. What is the present value of an annuity of \$96 a year, to continue 10 years, allowing compound interest at 6%?

4. The annual rent of a house is \$300. What would be the present value of the rents for 5 years, allowing compound interest at 7%?

5. What sum of ready money will purchase an annuity of $40, to continue 5 years at 5% compound interest?

Ans. *$173.18*

Test Questions.

421.—1. What is an *Imaginary Quantity?* Into what two factors may every imaginary quantity be resolved? How are the common rules for radicals modified for multiplication and division of imaginary quantities?

2. What does the *Binomial Theorem* express? What invariable laws govern the expansion of a binomial? How can the binomial formula be made to apply to raising a polynomial to any power?

3. What is a *Logarithm?* What is the base of common logarithms? What is the characteristic of a logarithm? What principles are given?

4. What is *Compound Interest?* What formula expresses the amount of any sum at compound interest for any number of years?

5. What is an *Annuity?* What is the formula for finding the amount of an annuity for any number of years at compound interest? For finding the present worth of an annuity for any number of years at compound interest?

SECTION LIII.

EXAMINATION PROBLEMS.

396. The following **Problems** may be used at the discretion of the teacher in testing the proficiency of pupils as they progress in the book.

The Articles in parentheses denote the portions of the text to which the problems relate.

(Articles 1-101.)

397.—Ex. 1. If $a=1$, $b=2$ and $c=3$, what is the value of $(a-b)^2+(b-c)^2+(c-a)^2$?

2. Add together $5x+3y-2z$, $-3x+2y-5z$, $2x-5y+3z$ and $-4x+4z$.

3. If $a=4$, $b=3$ and $c=2$, what is the value of $\sqrt{a^2+3b+c}$?

4. Subtract $-x-y-z$ from $x-y+z$.

5. Simplify the expression $(5a-7b+6c)+(5a-10b-20c)$.

6. Multiply $3x+2y-z$ by $x-2y+3z$, and prove the result by division.

7. If $a=1$, $b=2$, $c=3$ and $d=4$, find the value of $bc^2d-c^2bd+abcd+a^2(3bc-4b^2+5c^2)-a^3(7b-8d)$.

8. Divide $x^4-x^3y-xy^3+y^4$ by x^2+xy+y^2, and prove the result by multiplication.

9. Show that any quantity with a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.

10. Express as a fraction $m^2a^{-1}b^{-2}c$.

11. A vessel holding 120 gallons is partly filled by a spout which delivers 14 gallons in a minute; this is turned off, and a second spout, delivering 9 gallons in a minute, completes the filling of the vessel. How long did each spout run, the time occupied by both being 10 minutes?

(**Articles 103-209.**)

398.—Ex. 1. Show that the square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

2. Resolve $9a^2-6ac+c^2$ into two equal factors.

3. Show that the difference of any two equal powers of two quantities is divisible by the difference of the quantities.

4. Resolve a^3+x^3 into two factors.

5. What is the least common multiple of y^3-1 and y^2+x-2?

6. What is the greatest common divisor of x^2y-2xy, $2x^2-4x^2$ and $3x^2-6x$?

7. Show by an example that the greatest common divisor of two quantities is the product of all their common prime factors.

8. Required the sum of $\dfrac{x-a}{x^2-ax+a^2}$ and $\dfrac{1}{x+a}$.

9. What is the product of $\dfrac{x^n}{y^m}$ by $\dfrac{y^n}{x^m}$?

10. What is the quotient of $\dfrac{9x^2-4y^2}{x+y}$ divided by $\dfrac{3x-2y}{x^2-y^2}$?

11. A boy spends half a dollar more than half his money. Again, he spends half a dollar more than half his remaining money. A third time he does the same, and finds then his money all gone. How many dollars had he at first?

(Articles 210–291.)

399.—Ex. 1. Given $\frac{x}{2}+\frac{y}{3}=7$ and $\frac{x}{3}+\frac{y}{2}=8$, to find x and y.

2. Given $12x+8y+6z=1488$, $20x+15y+12z=2620$ and $30x+24y+20z=4560$, to find x, y and z.

3. Show that all the even powers of a negative quantity are positive and all the odd powers are negative.

4. What is the square of $-9x^{-1}y^{-3}$?

5. What is the square root of $x^4-2x^2y^2-2x^2+y^4+2y^2+1$?

6. Divide $\sqrt[3]{ab^2c^2}$ by $\sqrt[5]{a^2b^3c^4}$.

7. Given $\sqrt{x+19}=9-\sqrt{x+10}$, to find x.

8. A has two kinds of gold coins; 7 of the larger, together with 12 of the smaller, make \$288; and 12 of the larger, together with 7 of the smaller, make \$358. What is the value of each kind of coin?

9. A person has two horses and a saddle worth \$250. If the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value three times that of the first. What is the value of each horse?

(Articles 290–317.)

400.—Ex. 1. Given $4x^2-7=3x^2+9$, to find x.

2. Given $\frac{3x-4}{x-4}=9-\frac{x-2}{2}$, to find x.

3. Show that affected quadratic equations depend for their solution on the form of a squared binomial.

4. Given $x^4-13x^2=36$, to find x.

5. Given $x-14=4y$ and $4x-2y+y^2=11$, to find x and y.

6. The sum of the squares of two numbers is 706, and the difference of their squares is 544. What are the numbers?

7. A company for dining at a hotel had to pay $14.60. Had there been four less in the company, each person would have paid 60 cents more. Of how many persons did the company consist?

8. A and B sold 130 yards of cloth, of which 40 yards were A's and 90 B's, for $42. Now A sold for a dollar $\frac{1}{3}$ of a yard more than B did. How many yards did each sell for a dollar?

(Articles 319–363.)

401.—Ex. 1. Show that in any proportion the product of the extremes is equal to the product of the means.

2. Insert five arithmetical means between $\frac{1}{2}$ and $-\frac{1}{2}$.

3. Show that any number of geometrical means may be inserted between two given terms of a geometrical progression.

4. A board, $2\frac{1}{2}$ inches wide at the narrow end and 10 feet long, increases in width $1\frac{1}{2}$ inches for every foot in length. Required the width of the wider end.

5. Divide 49 into two such parts that the greater increased by 6 may be to the less diminished by 11 as 9 to 2.

6. Show that the mean proportional between two quantities is equal to the square root of their product.

7. During the late war, 594 men were raised from three towns, A, B and C, according to their populations. The population of A is to that of B as 3 to 5, and the population of B is to that of C as 8 to 7. How many men did each town furnish?

8. The product of two numbers is 63, and the square of their sum is to the square of their difference as 64 to 1. What are the numbers?

9. What is the value of the circulate .2323 . . . to infinity?

10. A servant worked twelve months, on the condition that for the first month's service he should receive 25 cents, for the second $1, for the third $4, and so on. What did his wages amount to?